The Harald Bohr Centenary

Proceedings of
a Symposium held in Copenhagen
April 24-25, 1987

Edited by C. BERG *and* B. FUGLEDE

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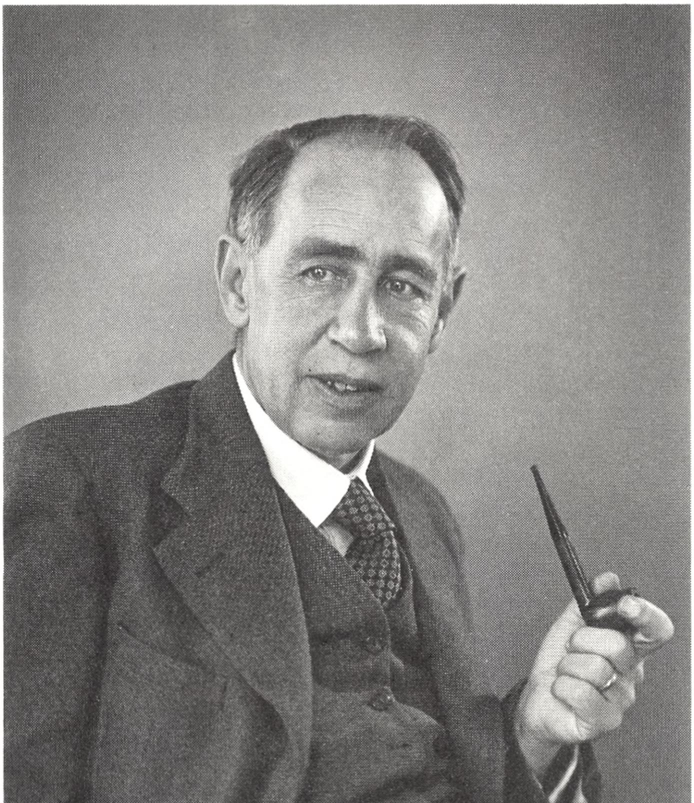

The Harald Bohr Centenary

Preface

These are the proceedings of the symposium held in Copenhagen, April 24-25, 1987, on the occasion of the centenary of Harald Bohr (April 22, 1887 – January 22, 1951). The symposium was arranged by the Danish Mathematical Society which appointed an organizing committee consisting of C. Berg, B. Fuglede, and L.-E. Lundberg. The scientific programme comprised 13 lectures of which 11 were given by invited foreign scholars. Former colleagues, students and friends of Harald Bohr profited from this occasion to commemorate the work and personality of a Danish mathematician of high international rank.

The organizing committee was happy to have the symposium opened by the rector of the University of Copenhagen, O. Nathan, a former student of Harald Bohr during those years of the war which they had to spend in Sweden.

The main achievement in the mathematical works of Harald Bohr – the theory of almost periodic functions – has developed in several directions during the past sixty years, and the papers in the present Proceedings will show some of these trends.

We take the opportunity of thanking all the participants for their presence and their scientific contribution to the symposium.

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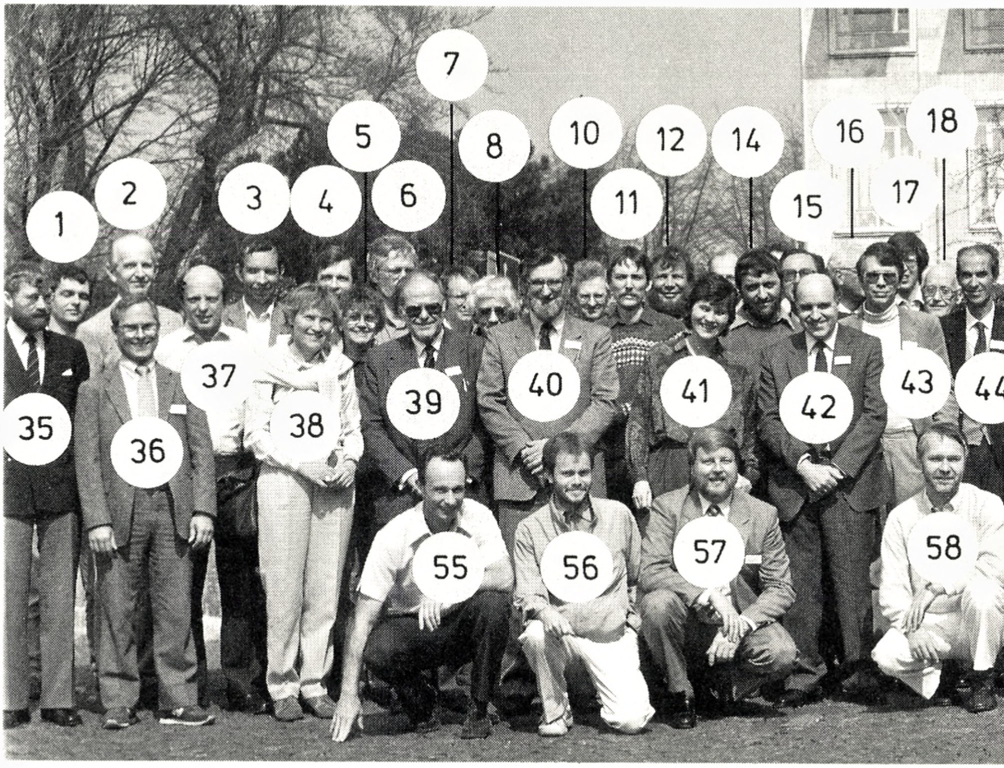
The Organizing Committee



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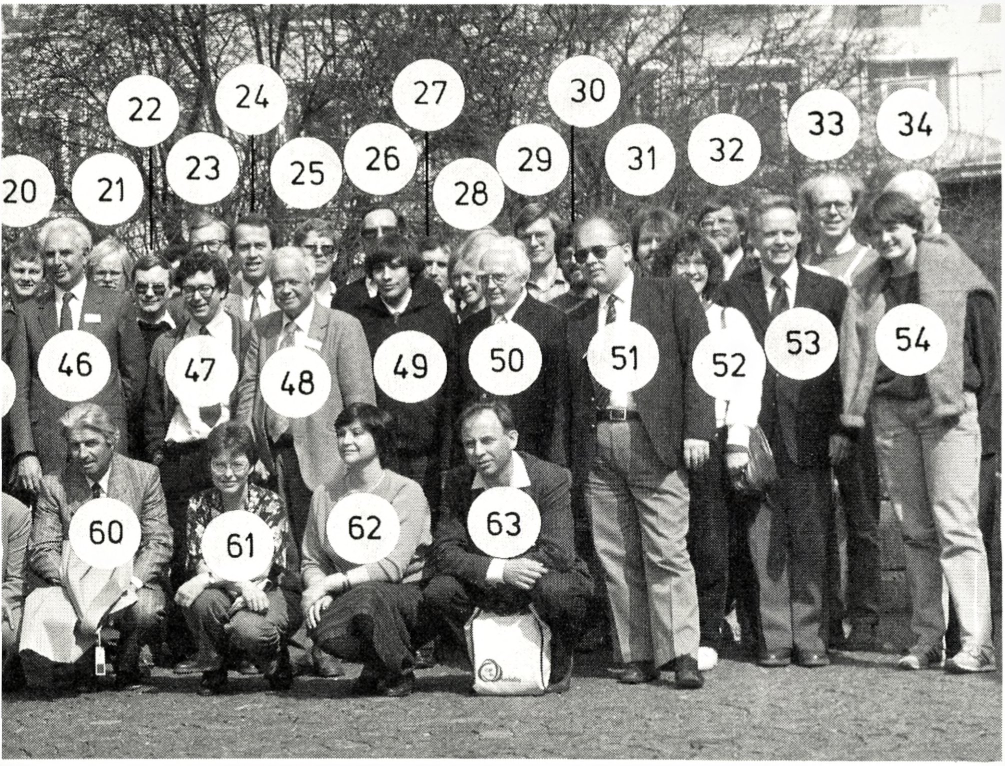

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Harald Bohr
Professor and Head of Department

By Hans Tornehave

This is an attempt to tell about Harald Bohr as we knew him, "we" meaning students and junior teachers. I was his student 1935–1940 and a junior teacher at the institute till after his death in 1951.

The nineteenhundred and thirties were the peak of Harald Bohr's career. The almost periodic functions had made him known all over the mathematical world. The original papers appeared in 1924–26 and the monograph in *Ergebnisse* in 1932. He had been a professor at the Technical University since 1914, but was called to our university in 1930. He got his new mathematical institute in 1934 at the 450$^{\text{th}}$ anniversary of the university. It was a grant from the Carlsberg Foundation, and it was built as a new wing of his brother's Institute for Theoretical Physics, which is now called the Niels Bohr Institute.

Before he started in his new position at the university, Harald Bohr went on a tour to U.S.A., where he visited Stanford University and the Institute for Advanced Study. He was a member of the Royal Danish Academy of Sciences and Letters from 1918, and he was the chairman of our Mathematical Society from 1935.

According to Harald Bohr the textbook in mathematical analysis known as "Bohr and Mollerup" was inspired by Jordan's *Cours d'Analyse,* but also Hardy: *A Course of Pure Mathematics* has had some influence. Johannes Mollerup has probably been somewhat underestimated and most of the details in the text-book have certainly been the result of discussions between the authors. However, Bohr wrote very much like he talked, and we who knew him hear the echo of his voice when we read his book. Hence, we think that Bohr is responsible for the formulation, but we also know that he was extremely willing to accept good ideas proposed by others.

Bohr did not give elementary lectures at the university. The students attended the lectures for engineering students on mathematical analysis, theoretical mechanics and physics, while they shared the chemistry lectures with the students of medicine. The students had separate elementary lectures only in geometry and astronomy.

In 1933 Bohr tried just once to lecture over his textbook for the mathematics students. The lecture went on until the fall term in 1935. Bohr was somewhat more sedate than in his younger days. He had given up the habit of keeping the sponge on the floor, kicking it ceilingwards and catching it neatly, when he had to erase something. He had been a top soccer player, expert dribbler and very popular. He continued playing even as a professor at a time when tailcoat and tophat were standard equipment at the university. In the thirties it was a lot more informal, although most students wore a suit and a tie at the lectures.

Bohr talked very fast and with a flat "a" reminiscent of Copenhagen dialect. He used the blackboard in a systematic way starting upper left and finishing lower right, writing in long horizontal lines. He supplied the lecture with many cross references while moving rapidly back and forth and underlining this and that on the blackboard in different colors. He did not spend much time on straightforward proofs, but where a trick was needed he demonstrated how the obvious method failed, and he also motivated the kind of trick to be used. He used many words in the text-book, but still more in his lectures.

For his advanced lectures Bohr always prepared a complete manuscript written by hand in a solid bound volume with ruled pages. If he had an interested assistant, he discussed the text with him, and suggestions from the assistant were quite often tried out in the manuscript and in the lecture. Nevertheless it happened quite often that Bohr improvised something, and his improvisations were the best parts of his lectures. He lectured regularly on number theory and on complex analysis, but the content of these lectures varied considerably. He arranged it usually so that his assistant could continue the lecture in a subsequent term and talk about his own particular interests.

Mathematicians of to-day would find Bohr's lectures rather old-fashioned, but one must remember the current state of mathematics at that time. The shift from combinatorial to algebraic topology had just started and Hilbert space and spectral theory were known, but hardly in the abstract form. General topology was also known, but it had not a quite definite form. Measure theory had not yet become abstract, and convexity theory was mostly finite dimensional.

Bohr was very open for new ideas and he enjoyed the abstract points of view, but he did not lecture on these modern subjects although he quite often treated very modern subjects in brief talks, e.g. in our mathematical society. We had also lectures by Bohr's friends, former pupils, young assistants etc. on such modern subjects, but before I start on this I must tell a little about the life at the institute.

The former institute building looks small to-day, but it was really quite ample. About 20 students per year started in the compulsory combination of mathematics, physics, chemistry and astronomy and about half of them specialized in mathematics. The other professors in mathematics were N. E. Nørlund and J. Hjelmslev, but Nørlund was also the director of the Geodetic Institute and only part time professor. He gave two lectures a week, sometimes on geodesy. Hjelmslev was really a genius and he contributed much to the understanding of the interactions of the axioms of geometry. His lectures were brilliant and convincing, but he did not really care about minor details, and many students found it hard to follow him. His assistant David Fog interpreted Hjelmslev's text quite well, and he was very popular with the students. Nørlund's assistant G. Rasch was well liked by the students who attended his exercises on differential equations, but he was dismissed some time in the thirties. He was son of a missionary and he became a preacher of statistics himself and did a lot to improve the statistical work in medicine and biology, and in the end he became a professor.

Most students of mathematics specialized with Bohr. His assistant J. Pál was also the mathematics teacher of the chemical engineering students. He was probably the first jewish mathematician helped by Harald Bohr. He was Hungarian, and Bohr found him in Göttingen, where he seemed more or less lost. He was interested in real and complex analysis, topology, convexity, formal algebra and projective geometry. He did not look like a jew, and very few persons knew that he came from a jewish family, and he did not really like jews in general. He was religious, and he called himself a catholic. He had very strict views on morality and on teaching, and he insisted that students should learn only what they were able to learn well. He was very helpful to several students including myself, but he could be very disagreeable to some students. His exercises in connection with Bohr's elementary lectures were considered a trial by most of the students.

It was unfortunate that there was too little contact between Bohr's students and the students who specialized with Hjelmslev or Nørlund, but it was fortunate that Bohr was the center of much debate and activity. Many teachers of the technical university participated in seminars arranged by Bohr and they lectured occasionally at the university. Among them were A. F. Andersen, Richard Petersen, Johannes Mollerup (who died in 1937), Kaj Rander Buch, Vilhelm Jørgensen, Svend Lauritzen. Most of all Børge Jessen, who stayed with us from 1941, and Svend Bundgaard, who was with us much of the time. Erik Sparre Andersen, Erling Følner and some more joined us towards the end of the thirties. A few high school teachers were also regular guests.

There were also mathematicians whom Bohr had helped to get away from Hitler's Germany. Most of them stayed in Denmark only for a short time, but Werner and Käte Fenchel came for good, and Otto Neugebauer was here for many years. He started our tradition of history of science. Olaf Schmidt and Asger Aaboe were his pupils.

The physicists had even more guests and there was fraternization between the two populations. Hevesy did chemistry and biology and talked Hungarian with Pál. Frisch and Meitner told us about their discoveries. The brothers Bohr talked much with each other, but always quite in private. It was very obvious that they were great friends. Occasionally, we also exchanged some small talk with Niels Bohr, but seldom with both brothers simultaneously.

An important event was the big lecture series in 1936/37 on almost periodic functions. Jessen lectured and Bohr was the most eager commentator. Most of the mathematicians mentioned above were in the audience and also some students. The lecture included the generalizations to Abelian groups and the theory of analytic almost periodic functions, but not the generalizations to Lebesgue-integrable functions. These were investigated very thoroughly shortly afterwards by Bohr and Følner in a large joint paper and in Følner's thesis. Følner and Jessen collected a nearly complete bibliography of almost periodic functions, and it was during this work that Følner found Bogoliubov's second proof of the approximation theorem.

It has been said that everybody in Hilbert's Göttingen discussed everything with everybody else, while nobody in Poincaré's Paris discussed anything with anybody else, and Bohr was as much influenced by Göttingen as Nørlund by Paris. Bohr was much attached to Landau, and he quoted occasionally some of Landau's deprecating remarks about Hilbert, but he really also admired Hilbert very much. It was probably one of the greatest disappointments in Bohr's life that nobody succeeded in finding a place for Landau outside Germany.

Hardy was another friend of Harald Bohr and had a lot of influence on him. Hardy was the typical diner at high table in college, who liked the learned discussions and enjoyed taking a standpoint and defending it, even in matters he knew little about. He was the genuine English combination of the extremely refined with the quite informal, and he was very outspoken. Bohr has told that Hardy called his English atrocious and that Hardy had to teach him that it was important to say "he did not come" rather than "he does not came". Hardy was obviously a clever teacher, and Bohr's English grew much better. His German was very efficient, but he spoke it with the flat Copenhagen "a".

It is easy to understand that Bohr and Hardy fascinated each other. Both liked taking standpoints on everything, but Bohr did it experimentally and his standpoints were to be changed eventually. Hardy enjoyed defending his standpoints as a kind of sport. Bohr and Hardy paid visits to each other and Hardy liked the Danish landscape with the red cows drawing circles in the pastures.

Our mathematical society had more frequent meetings in those days; there was no competition with advanced colloquia. We had a good many foreign guests. It is true that it was not very wealthy, but even a small grant went a long way. So, we were quite well informed about new mathematical events. For instance we had Landau's assistant Heilbronn giving a brief series of lectures on Vinogradov's proof of the weak Goldbach conjecture.

Bohr understood and accepted new ideas quite readily, as e.g. the theory of distributions when he heard Laurent Schwartz lecture on them shortly after the war. In his teaching, however, he stuck to the classical subjects, which he knew extremely well, but he encouraged the junior teachers to lecture on these modern subjects. Sv. Bundgaard lectured in abstract algebra and on Lebesgue integration theory. Occasionally a teacher from the technical university gave a course. Jessen has already been mentioned, but also Jakob Nielsen gave a course on his own subject, surface topology.

Theoretical logic was viewed by Bohr with some suspicion, and most of the other mathematicians at the institute agreed with him. Kronecker's strict point of view was generally respected, but Bohr and everybody else were as unwilling as Hilbert to abandon Cantor's paradise, and nobody was able to manage without Zermelo's result. In the teaching Landau's *Grundlagen der Analysis* was more or less chosen as a basis.

As a matter of fact the professor of philosophy Jørgen Jørgensen was a preacher of formal logic. He read an introductory course in philosophy. It was compulsory for all university students, but they could choose between three very different teachers. Jørgen

Jørgensen experimented with polyvalent logic, and he was also once invited to give a special course at the institute. One student specialized in logic, and Bohr was not very happy about it, but the student passed with nice marks. The attitude to logic changed while Jessen was head of department, and Gutmann Madsen started lecturing on it.

Bohr enjoyed talking informally with us when we were engaged in idle discussion in the lunch room at the institute, and when we were discussing whatever it might be, he quite often added some very surprising remarks. He was very observant and had a keen sense for all kinds of absurdities in the real world, and occasionally he enjoyed talking nonsense. I remember once, when the news of the discovery of the rabbits' "chewing pellets" first reached us, that one student stated that hares and rabbits were really ruminants, and, of course, he met intense opposition, but then Bohr appeared and he supported the student because, he said, he knew that these animals could not be imported in Sweden, and he thought that was because of the mouth and hoof disease, which attacked only ruminants. Then he went on telling that he once caught the mouth and hoof disease himself and it was really quite disagreeable.

Ulla Bohr has told a story from Bohr's visit in U.S.A. shortly after the crash in 1929. When they first visited a private home over there, Bohr went to the bathroom and got so much absorbed in studying the gadgets that the company became nervous and came looking for him. He was interested not only in things, however, but also in people and he had a deep understanding of relations between people.

Quite often Bohr celebrated the end of a term of lectures by inviting the participants and perhaps also some of his colleagues to a dinner in his home, and afterwards he might read something to us, and quite often something with overtones of absurdity. He has read to us from *Babbitt* by Sinclair Lewis and from *Winnie the Pooh* in the Danish translation "Peter Plys", and he made very similar comments on the two texts. Babbitt was the American who lived through the boom and the crash and who said and did just what everybody else said and did and understood nothing of what happened, and the ways of Winnie the Pooh were much the same, although his crash was less definite.

After the war Bohr became the Provost of Regensen, our closest equivalent of Trinity College. It is governed by three persons with mock clerical titles, the provost, the vice provost and the bell ringer, who is the students' representative. The provost lived in Regensen, where he had an old-fashioned, but comfortable apartment, and it was an attractive setting for his parties.

He also owned a fine old fisherman's cottage with leaded window-panes and a thatched roof. It is situated on a low cliff about half a mile south of Fynshav on the island Als. In those days the ferry harbour was at Mommark some 6 miles farther south and Fynshav was very peaceful. It was also on the edge of a very small village with a few farms and some small houses. Bohr invited his foreign and Danish friends to stay with him at Fynshav, and he found living quarters for them in the village. I visited him there a few times during the war, when he had no foreign guests, but it was charming to discover that the villagers knew Landau, Hardy, Weyl, Bochner and many other famous mathematicians.

We learned in Fynshav that Bohr was a nasty player of croquet and boccia, and on a rainy day the relatives of his mother, the Adler's, were even more nasty players of parlour games like writing lists of as many famous persons as possible with their last name starting with E. Occasionally, the blackboard was carried outside on the gravel and somebody gave a lecture. Some of Jacob Nielsen's results were presented there first. He had a house about half a mile northwest of Fynshav and the two families paid many visits to each other, Jacob Nielsen often travelling in his kayak.

Bohr's health was not quite satisfactory. On Als we saw how he spread sesam seeds on his oatmeal in the mornings, and during the terms he might take a little time off for recreation at Aldershvile by Bagsværd Lake, and we might have to go there to discuss some problem with him. Nevertheless he was always quite cheerful, although he was nervous now and then about the success of some effort to get somebody away from Germany.

We remember Harald Bohr as extremely mild mannered, but it would be very wrong to consider this as a symptom of weakness, and he could be incredibly stubborn when he fought for a cause that he felt was just. Once, when he had to judge a doctor's thesis, which was barely acceptable, he did accept it, but at its defence he told the doctor in no uncertain terms that it was just barely acceptable, and he did it in such a way that also the doctor was convinced.

As told above, Bohr enjoyed many kinds of absurdities, but he really hated the absurdities used by the German nazis as excuses for the worst atrocities. He also felt that the German jews should be helped indiscriminately, since they were persecuted indiscriminately. This led to his disagreement with Pál, who did not like jews indiscriminately, and Pál left the institute. Actually there were many jews among Pál's best friends, and the real background for the break was the strict religious-moralistic point of view of Pál, who wanted everybody to follow the straight path regardless of the kind of provocations they met with.

Harald Bohr died, and the message of death reached us at the institute a dreary winter morning immediately before we should start a day of examinations, which went off rather badly. But Bohr had been the kind of leader who left a healthy institute, which lived on and thrived. He would have enjoyed being with us to-day.

Matematisk Institut
Universitetsparken 5
DK-2100 København Ø
Denmark

Introduction to the Almost Periodic Functions of Bohr

By Christian Berg

0. *Introduction*

The content of this paper was presented at the Centenary of Harald Bohr with the purpose of serving as an introduction for the many non-specialists present. It is our hope that this written version will incourage the reader to study the work of Harald Bohr. The collected mathematical works appeared in 1952, cf. [8], and at the occasion of the Centenary his mathematical papers with a pedagogical aim – written in Danish – have been published, cf. [9].

In the following we will concentrate on Bohr's main results about almost periodic functions, but we shall briefly indicate how he was led to the theory and how it later merged into the theory of harmonic analysis on locally compact abelian groups. The so-called Bohr compactification of a group has become a standard concept in harmonic analysis.

The readers interested in a further study of almost periodic functions are referred to the many monographs on the subject, e.g. Amerio and Prouse [1], Besicovič [3], Bohr [7], Corduneanu [10], Maak [11]. A complete bibliography on almost periodic functions from 1923 to march 1987 has been collected, see [13].

1. *Background*

Harald and the two years older brother Niels were sons of the professor of physiology Christian Bohr, and from their youth they felt veneration for science and were acquainted with the scientists of the time. Harald began to study mathematics at the University of Copenhagen at the age of 17, and already in 1910 he defended his doctoral dissertation ([5]) on the summability theory of Dirichlet series, that is series of the form

$$\sum_{n=1}^{\infty} \frac{a_n}{n^z} = \sum_{n=1}^{\infty} a_n e^{-(\log n)z}, \tag{1}$$

where (a_n) is a sequence of complex coefficients, and $z = x+iy$ is a complex variable. Jensen had shown i 1884 that there is an abscissa of convergence γ_0 such that (1) is convergent for $x > \gamma_0$, divergent for $x < \gamma_0$.

Bohr showed that there is a decreasing sequence $\gamma_0 \geq \gamma_1 \geq \gamma_2 \geq$ of abscissas of summability such that (1) is Cesàro summable of order r for $x > \gamma_r$ but not for $x < \gamma_r$.

Furthermore, the width $w_r = \gamma_{r-1} - \gamma_r$ of the strip $\gamma_r < x < \gamma_{r-1}$, where the series is summable of order r but not of order $r-1$, satisfies

$$1 \geq w_1 \geq w_2 \geq \dots \quad . \tag{2}$$

Bohr could furthermore show that the inequalities (2) were characteristic for the sequence of summability abscissas because, for given numbers $\gamma_0 \geq \gamma_1 \geq \dots$ such that (2) holds, he constructed a Dirichlet series having these numbers as abscissas of summability. The sum of the series (1) is a holomorphic function f in the halfplane $x > \gamma_0$. By the Cesàro summability f has a holomorphic continuation to the half-plane $x > \lim_{r\to\infty} \gamma_r$. Bohr also showed the remarkable result that $\lim_{r\to\infty} \gamma_r$ is characterized as the infimum of the numbers α for which f has a holomorphic extension to the half-plane $x > \alpha$ satisfying an estimate

$$|f(x+iy)| \leq A + |y|^B,$$

where A,B depend on α.

About the same time the Hungarian mathematician Marcel Riesz had examined the summability theory of general Dirichlet series

$$\sum_{n=1}^{\infty} a_n e^{\lambda_n z}, \tag{3}$$

where (λ_n) is a sequence of real numbers. Bohr had also considered this general case, but in the dissertation he restricted the investigations to the special case of $\lambda_n = -\log n$.

As a result of his investigations on dirichlet series Bohr got into fruitful collaboration with Landau in Göttingen about the Riemann zeta function.

For a period of several years partially overlapping with the first world war Bohr was engaged in writing a treatise in Danish on mathematical analysis together with professor Mollerup. Bohr knew the famous Cours d'Analyse of Jordan from his years of study and he was very much influenced by it. The mathematical analysis textbook of Bohr and Mollerup should get an enormous influence on the teaching of mathematics in Denmark, and it was used from 1915 to the 1960'ies both at the University of Copenhagen and at the Technical University, although in revised editions. Further information about the life and work of Bohr can be found in his own lecture "Looking backwards" and in the memorial address by B. Jessen, both published in the collected mathematical works [8].

2. *Almost periodic functions*

It was after the completion of the mathematical analysis textbook that Bohr took up the investigations which should eventually lead to his main accomplishment, the theory of almost periodic functions. The starting point was an attempt to characterize the functions $f(z)$ which admit a representation by a Dirichlet series (3).

On a vertical line $z = x_0 + iy$ this leads to the representation of a function $f(x_0+iy)$ of a real variable y as sum of a series

$$\sum_{n=1}^{\infty} b_n e^{i\lambda_n y} \text{ where } b_n = a_n e^{\lambda_n x_0}.$$

Such series comprise Fourier series for periodic functions with period $p > 0$ corresponding to $\lambda_n = \frac{2\pi}{p} n,\ n \in \mathbf{Z}$. Bohr's main contribution was to give an intrinsic characterization of the class of functions $f: \mathbf{R} \to \mathbf{C}$ which can be uniformly approximated by *trigonometric polynomials,*

$$\sum_{n=1}^{N} a_n e^{i\lambda_n y}, \tag{4}$$

where the frequences λ_n can be arbitrary real numbers, and the coefficients a_n arbitrary complex numbers.

He proved that the uniform closure of the trigonomtric polynomials are those continuous functions which are *almost periodic* in a sense explained below.

If $f: \mathbf{R} \to \mathbf{C}$ is a function of a real variable and $\varepsilon > 0$, then $\tau \in \mathbf{R}$ is called a *translation number* or an *almost period* for f corresponding to ε if

$$|f(x+\tau) - f(x)| \leq \varepsilon \quad \text{for all} \quad x \in \mathbf{R}.$$

A subset $A \subseteq \mathbf{R}$ is called *relatively dense* in $\mathbf{R}$, if there exists a sufficiently big number $l > 0$ such that every interval of length l contains at least one number from A.

Finally a continuous function $f: \mathbf{R} \to \mathbf{C}$ is called *almost periodic,* if for every $\varepsilon > 0$ the set $\{\tau_f(\varepsilon)\}$ of translation numbers for f corresponding to ε is relatively dense.

In other words, a continuous function *f is almost periodic if to every $\varepsilon > 0$ there corresponds a number $l = l(\varepsilon) > 0$ such that any interval of length l contains at least one number τ such that*

$$|f(x+\tau) - f(x)| \leq \varepsilon \quad \textit{for all} \quad x \in \mathbf{R}.$$

A continuous periodic function is almost periodic since a period p is a translation number corresponding to any $\varepsilon > 0$. If f is an almost periodic function which is non-periodic, and if $l(\varepsilon)$ denotes the smallest possible length corresponding to $\varepsilon > 0$, then $l(\varepsilon)$ will increase to infinity as ε decreases to zero. In fact if $l(\varepsilon) \leq l$ for all $\varepsilon > 0$, then the interval $[l,2l]$ contains a sequence (τ_n) such that τ_n is a translation number corresponding to $\frac{1}{n}$. Any accumulation point for the sequence (τ_n) is a period for f.

The first basic result in the theory is easy to prove: An almost periodic function is *uniformly continuous and bounded.*

The set $\mathcal{AP}$ of almost periodic functions is stable under addition and multiplication, so $\mathcal{AP}$ is an algebra of functions. More generally if $f_1,\ldots,f_n : \mathbf{R} \to \mathbf{C}$ are almost periodic and $\varphi : \mathbf{A} \to \mathbf{C}$ is a continuous function *defined on a subset* $\mathbf{A} \subseteq \mathbf{C}^n$ such that

$$\text{closure}\ \{(f_1(x),\ldots,f_n(x))\ |x \in \mathbf{R}\} \subseteq \mathbf{A},$$

then $\varphi(f_1(x),\ldots,f_n(x))$ is again almost periodic.

This is not so obvious and uses the fact that there exists for every $\varepsilon > 0$ a relatively dense set of *common* translation numbers for $f_1,\ldots,f_n$ corresponding to ε.

The principal concept for the further development of the theory is the *mean value* of an almost periodic function f. Bohr proved that the number

$$\frac{1}{T}\int_a^{a+T} f(x)dx$$

has a limit as T tends to infinity, even uniformly for $a \in \mathbf{R}$. This limits is called the mean value of f and is denoted $\mathcal{M}\{f\}$.

It is easy to see that $\mathcal{M}$ is a positive linear functional on $\mathcal{AP}$, and if $f \geq 0, f \neq 0$ then $\mathcal{M}\{f\} > 0$. If we put

$$(f,g) = \mathcal{M}\{f\bar{g}\} \quad \text{for} \quad f,g \in \mathcal{AP},$$

then $(\cdot,\cdot)$ is a scalar product, turning $\mathcal{AP}$ into a pre Hilbert space with the norm $\|f\| = \sqrt{(f,f)}$. The exponentials e_λ, $\lambda \in \mathbf{R}$ defined by $e_\lambda(x) = e^{i\lambda x}$ form an orthonormal family so $\mathcal{AP}$ is a non-separable pre Hilbert space. It is not complete.

With $f \in \mathcal{AP}$ Bohr associated the orthogonal expansion

$$f \sim \sum_{\lambda \in \mathbf{R}} a_\lambda e^{i\lambda x}, \tag{5}$$

where $a_\lambda = (f,e_\lambda) = \mathcal{M}\{f(x)e^{-i\lambda x}\}$.

Sometimes $\lambda \to a_\lambda$ is called the *Bohr transform* of f. For any finite set Λ of real numbers Bessel's approximation theorem yields

$$\|f\|^2 = \|f - \sum_{\lambda \in \Lambda} a_\lambda e_\lambda\|^2 + \sum_{\lambda \in \Lambda} |a_\lambda|^2, \tag{6}$$

showing that only countably many of the numbers a_λ, $\lambda \in \mathbf{R}$ are different from zero. Therefore, the orthogonal expansion (5) has only countably many non-zero terms; it is called the *(almost periodic) Fourier series of f*. The set $S = \{\lambda \in \mathbf{R} | a_\lambda \neq 0\}$ is called the *spectrum of f*, and the numbers $\lambda \in S$ are called the *frequences* of f.

It is furthermore easy to see that the Fourier series of a periodic function f coincides with the almost periodic Fourier series of f.

The theory developed so far is quite elementary. The importance of the theory was underlined by the following fundamental results, the proofs of which given by Bohr were long and difficult.

The theorems are:

(A) *The uniqueness theorem.*
If $f,g \in \mathcal{AP}$ have the same Fourier series then $f = g$. Equivalently $(e_\lambda)_{\lambda \in \mathbf{R}}$ is a maximal orthonormal system in $\mathcal{AP}$.

(B) *Parseval's formula.*

$$\|f\|^2 = \sum_{\lambda \in \mathbf{R}} |a_\lambda|^2 \quad \text{for any} \quad f \in \mathcal{AP}.$$

(C) *The approximation theorem.*
For $f \in \mathcal{AP}$ and $\varepsilon > 0$ there exists a trigonometric polynomial p of the form (4) *such that $|f(x) - p(x)| \leq \varepsilon$ for all $x \in \mathbf{R}$.*

The theory outlined so far appeared in two long papers in Acta Mathematica from 1924 and 1925, see [6],I,II, comprising more than 200 pages. The results had been announced in two notes in Comptes Rendus de l'Academie des Sciences, Paris 1923, see [8].

The first Acta paper contains the proof of Theorem B, and Theorem A is an easy consequence of Theorem B. In the proof of Theorem B Bohr considered for $T > 0$ the piecewise continuous function f_T which is equal to f on $[0, T[$ and periodic with period T. By Parseval's formula for periodic functions one has

$$\frac{1}{T}\int_0^T |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |a_{n,T}|^2,$$

where

$$a_{n,T} = \frac{1}{T}\int_0^T f(x) e^{-in\frac{2\pi}{T}x}\, dx.$$

Via a very delicate analysis Bohr obtained the result by letting $T \to \infty$. In the second Acta paper Bohr proved the approximation theorem using periodic functions of infinitely many variables.

In 1927 Bochner gave the following very important characterization of almost periodic functions, cf. [4]:

A function f: $\mathbf{R} \to \mathbf{C}$ is almost periodic if and only if it is continuous and the set of translates $\{f(x+a) \mid a \in \mathbf{R}\}$ has compact closure in the uniform metric.

The importance of this result lies in the fact that the compactness characterization can be used as starting point for the more general theory of almost periodic functions on groups as developed by von Neumann in 1934. From Bochner's result it is also obvious that the sum and product of almost periodic functions are again almost periodic.

Alternative proofs of the three fundamental theorems A, B, C were given shortly after Bohr's work by many different mathematicians e.g. Bochner, de la Vallée Poussin, Weyl and Wiener. This demonstrates the enormous interest the theory raised.

In a third major paper in Acta Mathematica from 1926 ([6],III) Bohr studied analytic almost periodic functions and their corresponding Dirichlet series.

For the definition of this concept it is useful to introduce the notion of an *equi-almost periodic* family $\mathcal{F}$ of continuous functions f: $\mathbf{R} \to \mathbf{C}$, thereby meaning that the set of common translation numbers for the functions in $\mathcal{F}$ corresponding to $\varepsilon > 0$ is relatively dense, i.e.

$$\bigcap_{f \in \mathcal{F}} \{\tau_f(\varepsilon)\} \quad \text{is relatively dense for any } \varepsilon > 0.$$

An analytic function f in a vertical strip $\alpha < x < \beta$ in the complex plane is called almost periodic in the strip if the family $\mathcal{F} = \{f(x+iy) \mid x \in]\alpha,\beta[\}$ is equi-almost periodic as functions of $y \in \mathbf{R}$. It turns out that the functions in $\mathcal{F}$ have the same frequences (λ_n) and that the Fourier coefficients

$$a_n(x) = \mathcal{M}_y \{f(x+iy)e^{-i\lambda_n y}\}$$

have the form $a_n e^{\lambda_n x}$ for a constant $a_n \neq 0$, showing that the Fourier expansion has the form

$$f(x+iy) \sim \Sigma\, a_n e^{\lambda_n (x+iy)}$$

called the *Dirichlet expansion* of f.

We shall not go further into the analytic almost periodic functions, which in a sense was the goal of Bohr's investigations.

3. *The Bohr compactification*

Let us consider the theory from another point of view.

The continuous group characters of the real line, i.e. the continuous homomorphisms of $(\mathbf{R},+)$ into $(\mathbf{T},\cdot)$, where

$$\mathbf{T} = \{z \in \mathbf{C} \mid |z| = 1\},$$

are precisely the functions $(e_\lambda)_{\lambda \in \mathbf{R}}$. The coarsest topology on $\mathbf{R}$ for which these functions e_λ, $\lambda \in \mathbf{R}$ are continuous, is strictly coarser than the ordinary topology. We propose to call it the *Bohr topology*. With the Bohr topology the real line is organized as a topological group, and a basis for the neighbourhoods of zero is given by the following sets

$$[\lambda_1,\dots,\lambda_m;\delta] = \{\tau \in \mathbf{R} \mid |e^{i\lambda_1\tau}-1| < \delta,\dots,|e^{i\lambda_m\tau}-1| < \delta\},$$

where $m \in \mathbf{N}$, $\lambda_1,\dots,\lambda_m \in \mathbf{R}$ and $\delta > 0$ are arbitrary.

The real line with the Bohr topology is not compact, not even locally compact, but it can be compactified. Let $\mathbf{T}_\lambda$ be a copy of the circle group for each $\lambda \in \mathbf{R}$ and let

$$j : \mathbf{R} \to \prod_{\lambda \in \mathbf{R}} \mathbf{T}_\lambda$$

be defined by

$$j(x) = (e_\lambda(x))_{\lambda \in \mathbf{R}} = (e^{i\lambda x})_{\lambda \in \mathbf{R}}.$$

The product set is a compact group under the product topology. The mapping j is clearly a homeomorphism of $\mathbf{R}$ with the Bohr topology onto the image $j(\mathbf{R})$. The closure of $j(\mathbf{R})$ is a compactification of $\mathbf{R}$ with the Bohr topology, called the *Bohr compactification* of $\mathbf{R}$ and denoted $\beta(\mathbf{R})$, i.e.

$$\beta(\mathbf{R}) = \overline{j(\mathbf{R})},$$

which is a compact group. In the sequel we identify $\mathbf{R}$ and $j(\mathbf{R})$.

By the approximation theorem an almost periodic function $f\colon \mathbf{R} \to \mathbf{C}$ is uniformly continuous in the Bohr topology, and therefore it has a unique continuous extension F to the Bohr compactification. Conversely, if $F\colon \beta(\mathbf{R}) \to \mathbf{C}$ is a continuous function on the compact group $\beta(\mathbf{R})$, then it is uniformly continuous, and so is the restriction f of F to the real line with the Bohr topology. This means that for any $\varepsilon > 0$ there exists a neighbourhood of zero of the form $[\lambda_1,\dots,\lambda_m;\delta]$ such that

$$|f(x+\tau)-f(x)| \le \varepsilon \quad \text{for all} \quad \tau \in [\lambda_1\dots,\lambda_m;\delta],$$

but this set is an ordinary neighbourhood of zero and relatively dense as is easily seen, so f is actually almost periodic.

This shows that there is a one-to-one correspondence between the almost periodic functions of Bohr and the continuous functions on the Bohr compactification $\beta(\mathbf{R})$.

The Bohr compactification $\beta(\mathbf{R})$ can be described as the set of all characters of $\mathbf{R}$, i.e. the set of all homomorphisms $\varphi\colon \mathbf{R} \to \mathbf{T}$. In fact, since $\beta(\mathbf{R})$ is the closure of the set of continuous characters, $\beta(\mathbf{R})$ consists of characters, and the fact that all characters belong to $\beta(\mathbf{R})$ is an easy consequence of Kroneckers's theorem.

4. *Harmonic analysis on locally compact abelian groups*

Bohr's theory of almost periodic functions has many resemblances with those of Fourier series and Fourier integrals. During the 1930'ies these three theories merged into a common theory called harmonic analysis on locally compact abelian groups. Many mathematicians contributed to this achievement e.g. Bochner, van Kampen, Pontryagin, Weil. The starting point was the theorem of Haar about the existence of an invariant measure on a locally compact group, now called Haar measure. With the publication in 1940 of Weil's fundamental monograph [12] the theory became widely known although many simplifications and refinements have appeared since then.

To every locally compact abelian group G is associated a *dual group* $\hat{G}$. As a set $\hat{G}$ concists of the continuous characters of G, i.e. the continuous homomorphisms $\gamma\colon G \to \mathbf{T}$. With pointwise multiplication and the topology of uniform convergence on compact subsets of G it turns out that $\hat{G}$ is a locally compact abelian group. It is customary to write (x,γ) in place of $\gamma(x)$ for $x \in G$, $\gamma \in \hat{G}$.

For a continuous function $f\colon G \to \mathbf{C}$ with compact support the Fourier transform $\hat{f}\colon \hat{G} \to \mathbf{C}$ is defined by

$$\hat{f}(\gamma) = \int f(x)\,\overline{(x,\gamma)}\; dm_G(x) \text{ for } \gamma \in \hat{G},$$

and it is possible to choose the Haar measures m_G and $m_{\hat{G}}$ on G and $\hat{G}$ in such a way that

$$\int_G |f(x)|^2 dm_G(x) = \int_{\hat{G}} |\hat{f}(\gamma)|^2 dm_{\hat{G}}(\gamma) \tag{7}$$

for all such f. This formula shows that the Fourier transformation $f \to \hat{f}$ has a unique extension to an isometry of $L^2(G)$ onto $L^2(\hat{G})$.

For $G = \mathbf{T}$ we have $\hat{G} \approx \mathbf{Z}$ and $\hat{f}(n)$ is the n'th Fourier coefficient, while (7) is Parseval's formula.

For $G = \mathbf{R}$ we have $\hat{G} \approx \mathbf{R}$, $\hat{f}$ is the ordinary Fourier transform and (7) is Plancherel's theorem.

For $G = \beta(\mathbf{R})$ and an almost periodic function $f\colon \mathbf{R} \to \mathbf{C}$ with unique continuous extension $F\colon \beta(\mathbf{R}) \to \mathbf{C}$ to the Bohr compactification $\beta(\mathbf{R})$, it turns out that

$$\mathcal{M}\{f\} = \int F dm_{\beta(\mathbf{R})} ,$$

i.e. the mean value of f is the Haar integral of the extension F. The dual group of $\beta(\mathbf{R})$ can be identified with $\mathbf{R}$ with the discrete topology, and $\hat{F}(\lambda) = a_\lambda$, the λ'th Fourier coefficient, while (7) is Parseval's formula, cf. (B) in §2.

Pontryagin's duality theorem states that the dual group of $\hat{G}$ can be identified with G, i.e. $\hat{\hat{G}} \approx G$.

Furthermore, for any locally compact abelian group G there is a Bohr compactification $\beta(G)$, which can be realized as the compact dual group of $\hat{G}$ considered as a discrete group. Again there is a one-to-one correspondence between continuous almost periodic functions on G and continuous functions on $\beta(G)$. The term Bohr compactification seems to have been introduced by Anzai and Kakutani in two papers from 1943, cf. [2].

5. *Conclusion*

We shall not attempt to describe the many generalizations and applications of the theory of almost periodic functions. The literature is enormous, cf. [13], and it would be an overwhelming task.

The other papers in this volume will shed some light on the various aspects of the subject and thereby show the richness and beauty of the theory initiated by Harald Bohr.

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Copies of the bibliography for individual use are available from A. M. Fink, Mathematics Department, Iowa State University, Ames, Iowa 50011.

Matematisk Institut
Universitetsparken 5
2100 København Ø
Denmark

Almost-Periodic Functions in Banach Spaces

By Luigi Amerio

1. *Definition of almost-periodic function. Elementary properties*

The general theory of almost-periodic (a.p.) functions with complex values, created by Harald Bohr in his two classical papers published in Acta Mathematica in 1925 and 1926 [1], has been greatly developed by Weil, De La Vallée-Poussin, Bochner, Stepanov, Wiener, Bogoliubov, Levitan.

Fundamental results, in the theory of a.p. linear ordinary differential equations, are expressed by the theorems of Bohr-Neugebauer and of Favard [2].

Bohr's theory was then, in a particular case, extended by Muckenhaupt [3] and, subsequently, by Bochner [4] and by Bochner and von Neumann [5] to very general abstract spaces.

The extension to Banach spaces has, in particular, revealed itself of great interest, in view of the fundamental importance of these spaces in theory and applications.

Let X be a Banach space; if $x \in X$, we shall indicate by $\|x\|$, or by $\|x\|_X$, the corresponding norm.

Let J be the interval $-\infty < t < +\infty$ and

$$x = f(t) \tag{1.1}$$

a *continuous function* (in the *strong sense*), defined on J and with values in X.

When t varies in J the point $x = f(t)$ describes, in the X space, a set which is called the range of the function $f(t)$, indicated by R_f.

A set $E \subset J$ is said to be *relatively dense* (r.d.) if there exists a number $l > 0$ *(inclusion length)* such that every interval $[a, a+l]$ contains at least one point of E.

We shall now say that *the function $f(t)$ is almost-periodic (a.p.) if to every $\varepsilon > 0$ there corresponds an r.d. set $\{\tau\}_\varepsilon$ such that*

$$\sup_J \|f(t+\tau) - f(t)\| \leq \varepsilon \qquad \forall \tau \in \{\tau\}_\varepsilon . \tag{1.2}$$

Each $\tau \in \{\tau\}_\varepsilon$ is called an *ε-almost period* of $f(t)$; to the set $\{\tau\}_\varepsilon$ therefore corresponds an *inclusion length* l_ε and it is clear that, when $\varepsilon \to 0$, the set $\{\tau\}_\varepsilon$ becomes rarified, while (in general) $l_\varepsilon \to +\infty$.

The above definition was given by Bochner and is an obvious extension of the definition adopted by Bohr for his theory of a.p. functions. It is, undoubtedly, in itself a very significant definition: its real depth can actually be understood only "a posteriori", from the beauty of the theory constructed on it and the importance of its applications.

The theory of a.p. functions with values in a Banach space is, in the way it is treated by Bochner, similar to Bohr's theory of numerical a.p. functions: new developments arise, as is natural, in connection with questions on compactness and boundedness. These questions (which have been particularly studied in Italy) are of notable interest in the integration of a.p. functions and, more generally, in the integration of abstract a.p. partial differential equations [6].

Let us now recall the first properties of a.p. functions, which can be easily deduced from their definition.

We add that when we say that $f(t)$ is uniformly continuous, or bounded, or that the sequence $\{f_n(t)\}$ converges uniformly etc., we always mean that this occurs on the whole interval J.

I *$f(t)$ a.p. $\Rightarrow f(t)$ uniformly continuous (u.c.).*

II *$f(t)$ a.p. $\Rightarrow R_f$ relatively compact (r.c.)* (that is the closure $\bar{R}_f$ is compact).

III *$f_n(t)$ a.p. $(n = 1, 2,\ldots)$, $f_n(t) \to f(t)$ uniformly $\Rightarrow f(t)$ a.p.*

IV *$f(t)$ a.p., $f'(t)$ uniformly continuous $\Rightarrow f'(t)$ a.p.*

V *$x = f(t)$ X-a.p., $y = g(x)$ with values in Y* (Banach) *and continuous on $\bar{R}_f \Rightarrow g(f(t))$ Y-a.p.*

In particular:

$$f(t) \text{ a.p.}, k>0 \Rightarrow \|f(t)\|^k \text{ a.p.}$$

2. *Bochner's criterion*

The class of a.p. functions has been characterized by Bochner by means of a compactness criterion, which plays an essential role in the theory and in applications. The starting point consists in considering, together with a given function $f(t)$, the set of its translates $\{f(t + s)\}$ and its closure $\overline{\{f(t + s)\}}$ with respect to uniform convergence. We have then:

VI *Let $f(t)$ be continuous, from J to X. A necessary and sufficient condition for $f(t)$ to be a.p. is that from every sequence $\{s_n\}$ it may be possible to extract a subsequence $\{l_n\}$ such that the sequence $\{f(t + l_n)\}$ be uniformly convergent.*

A very important consequence of Bochner's criterion is that *the sum $f(t) + g(t)$ of two X - a.p. functions is X - a.p.; the product $\varphi(t)f(t)$ of $f(t)$, X - a.p., by a numerical a.p. function $\varphi(t)$, is a.p.* It follows, in particular, the almost-periodicity of all *trigonometric polynomials:*

$$P(t) = \sum_{1}^{n} a_k e^{i\lambda_k t} \quad (a_k \in X, \lambda_k \in J).$$

Observation. Let $x = f(t) \in L^p_{\mathrm{loc}}(J; X)$, with $1 \leq p < +\infty$: assume in other words, that $\int_\Delta \|f(t+\eta)\|^p d\eta < +\infty \quad \forall t \in J$, where $\Delta = [0,1]$.

The function $f(t)$ is said to be *a.p. in the sense of Stepanov if to every* $\varepsilon > 0$ *there corresponds an r.d. set* $\{\tau\}_\varepsilon$ *such that*

$$\sup_J \left\{ \int_\Delta \|f(t+\tau+\eta) - f(t+\eta)\|^p d\eta \right\} \leq \varepsilon \quad \forall \tau \in \{\tau\}_\varepsilon. \tag{1.3}$$

As has been observed by Bochner, *the almost-periodicity in the sense of Stepanov can be reduced to that in the sense of Bohr (for vector valued functions).* Consider, in fact, the Banach space $L^p(\Delta; X)$ and define, $\forall t \in J$, the vector $\tilde{f}(t) = \{f(t+\eta)\} \in L^p(\Delta; X)$. We have then

$$\left\{ \int_\Delta \|f(t+\tau+\eta) - f(t+\eta)\|^p d\eta \right\}^{1/p} = \|\tilde{f}(t+\tau) - \tilde{f}(t)\|_{L^p(\Delta; X)}$$

and the thesis follows from (1.3).

3. *Harmonic analysis of almost-periodic functions*

The harmonic analysis of a.p. functions extends to these the theory of Fourier expansions of periodic functions. The following statements hold:

VII *(approximation theorem). If $f(t)$ is a.p. there exists, $\forall \varepsilon > 0$, a trigonometric polynomial $P_\varepsilon(t)$ such that*

$$\sup_J \|f(t) - P_\varepsilon(t)\| \leq \varepsilon.$$

VIII *(theorem of the mean). If $f(t)$ is a.p. there exists the mean value*

$$M(f(t)) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)\, dt$$

It follows that the function of λ

$$a(\lambda; f) = M(f(t) e^{-i\lambda t})$$

is defined on J; $a(\lambda; f)$ takes its values in X, as does $f(t)$: we shall call this function the *Bohr transform* of the a.p. function $f(t)$.

It can be seen, by VII, that $a(\lambda; f) = 0$ *on the whole of* J, *with the exclusion, at most, of a sequence* $\{\lambda_n\}$.

The values λ_n for which $a_n = a(\lambda_n; f) \neq 0$ are called the *characteristic* exponents of $f(t)$. The vectors a_n are the *Fourier coefficients of* $f(t)$, to which we can associate the *Fourier series*

$$f(t) \sim \sum_{1\ n}^{\infty} a_n e^{i\lambda_n t}.$$

IX *(uniqueness theorem)*

$$f(t) \text{ and } g(t)\ X\text{-a.p.},\ a(\lambda; f) \equiv a(\lambda; g) \Rightarrow f(t) \equiv g(t).$$

The *correspondence* between *almost-periodic functions* and their *Bohr transforms* is therefore *one-to-one*. A property of the transform $a(\lambda; f)$ is given by the following proposition:

X $a(\lambda'; f) = 0 \Rightarrow \lim_{\lambda \to \lambda'} a(\lambda; f) = 0$, *that is the Bohr transform is continuous at all points in which it vanishes. Furthermore:*

$$\lim_{\lambda \to \infty} a(\lambda, f) = 0 \quad , \quad \lim_{n \to \infty} a_n = 0,$$

and, for Hilbert spaces:

$$M(\|f(t)\|^2) = \sum_{1\ n}^{\infty} \|a_n\|^2 \qquad \text{(Parseval's equality)}.$$

We recall moreover that *Bochner's approximation polynomial* can be constructed also in the abstract case.

4. *Weakly almost-periodic functions*

Given the Banach space X, we shall call X^* its dual space (a Banach space too) constituted by the linear functionals continuous on X. If $x \in X$, $x^* \in X^*$, we shall indicate by $<x^*, x>$ the complex value that, through the functional x^*, corresponds to x, and by $\|x^*\|$ the norm of x^*.

We shall say that $f(t)$, with values in X, is *weakly almost periodic (w.a.p.) if,* $\forall x^* \in X^*$, *the numerical function*

$$<x^*, f(t)>$$

is a.p. [7].

As may be seen, the definition given here has, with respect to that of an *a.p.* function, the same relation as the definition of weakly continuous function has with respect to that of continuous function.

Its interest is particularly connected with statement XIV below. A different definition of weak almost periodicity is due to Eberlein [8]: the *w.a.p.* functions in the sense of Eberlein possess notable properties, particularly in relation to ergodic theorems.

It is clear (as $<x^*, x> \leq \|x^*\|\,\|x\|$) that $f(t)$ *a.p.* $\Rightarrow f(t)$ *w.a.p.* In order to indicate that $\{x_n\}$ is a sequence *converging weakly* to x (i.e. $<x^*, x_n> \rightarrow <x^*, x>$, $\forall x^* \epsilon X^*$) we shall make use of all the following notations:

$$x_n \xrightarrow{*} x, \text{ or } \lim_{n\to\infty}{}^{*} x_n = x,$$

and x is called the weak limit (which, if it exists, is also unique) of the sequence $\{x_n\}$. Let us remember that, in an arbitrary Banach space, a sequence $\{x_n\}$ can be *scalarly convergent* (i.e. $\lim <x^*, x_n>$ exists and is finite $\forall x^* \epsilon X^*$) without necessarily being weakly convergent, that is without there being an x which is its weak limit. If this circumstance is not present (i.e. if scalar convergence implies weak convergence) the space X is said to be *semicomplete* (reflexive, and, in particular, Hilbert spaces are semicomplete).

Let us now indicate some properties of *w.a.p.* functions.

XI *$f(t)$ w.a.p. $\Rightarrow R_f$ bounded and separable.*

When necessary, we can therefore assume that X is separable.

XII *$f_n(t)$ w.a.p. $(n = 1, 2, \ldots)$, $f_n(t) \xrightarrow{*} f(t)$ uniformly $\Rightarrow f(t)$ w.a.p.* ($f_n(t) \xrightarrow{*} f(t)$ uniformly means that, $\forall x^* \epsilon X^*$, $<x^*, f_n(t)> \rightarrow <x^*, f(t)>$ *uniformly).*

XIII *Let X be semicomplete and $f(t)$ weakly continuous. Then $f(t)$ w.a.p. $\Leftrightarrow \forall \{s_n\}$ there exists a subsequence $\{s'_n\}$ such that $\{f(t + s'_n)\}$ is uniformly weakly convergent.*

This proposition extends Bochner's criterion to *w.a.p.* functions (though with a restrictive hypothesis on the nature of the space X).

As we have already observed, $f(t)$ *a.p.* $\Rightarrow f(t)$ *w.a.p.* It is important to note that the property that has to be added to weak almost-periodicity to obtain almost-periodicity is one of *compactness*. The following theorem can, in fact, be proved.

XIV *$f(t)$ w.a.p. and R_f r.c. $\Rightarrow f(t)$ a.p.*

5. *Integration of almost-periodic functions*

If $f(t)$ is an *a.p.* function with values in a Banach space X, we will write, in what follows,

$$F(t) = \int_0^t f(\eta)\, d\eta. \tag{5.1}$$

The problem of the integration of *a.p.* functions in Banach spaces is of notable interest, also because it serves, so to say, as a *model* for classifying Banach spaces in relation to the theory of abstract *a.p.* equations.

If X is Euclidean, then Bohr's theorem holds: $F(t)$ bounded $\Rightarrow$ $F(t)$ a.p.

For the *general case (X arbitrary Banach space), the almost-periodicity of $F(t)$ has been proved by Bochner under the hypothesis that R_F is r.c.*

This condition is obviously much more restrictive than that of boundedness; it can not however be substituted in the general case by the latter, as can be shown in the following example (Amerio, [9]).

Consider, in fact, the space l^∞ of bounded sequences of complex numbers: $x = \{\xi_n\}$, with $\|x\| = \operatorname{Sup}_n |\xi_n|$. The function $f(t) = \{n^{-1} \cos(t/n)\}$ is *a.p.* and has the integral $F(t) = \{\sin(t/n)\}$, which is *bounded* ($\|F(t)\| \leq 1$) and *weakly a.p.* (see a) below), but *not a.p.*

One can prove nevertheless [9] that *Bohr's enunciation remains unaltered if the space X is uniformly convex* (it holds therefore in *Hilbert spaces*, in l^p and L^p, with $1 < p < +\infty$).

Let us prove now the following theorems.

XV (Bochner) *X arbitrary, $f(t)$ a.p., R_F r.c.* $\Rightarrow$ *$F(t)$ a.p.*
XVI (Amerio) *X uniformly convex, $f(t)$ a.p., $F(t)$ bounded* $\Rightarrow$ *$F(t)$ a.p.*

a) Proof of the theorem XV. As R_F is r.c., $F(t)$ is bounded:

$$\operatorname{Sup}_J \|F(t)\| = M < +\infty. \tag{5.2}$$

Furthermore, $\forall x^* \in X^*$,

$$|\langle x^*, F(t)\rangle| = |\langle x^*, \int_0^t f(\eta)\, d\eta\rangle| = |\int_0^t \langle x^*, f(\eta)\rangle d\eta| \leq \|x^*\|\, M.$$

As $\langle x^*, f(t)\rangle$ is *a.p.*, from Bohr's theorem it follows that $\langle x^*, F(t)\rangle$ is *a.p.*; $F(t)$ is therefore *w.a.p.*

R_F has been supposed r.c.; our thesis follows then from theorem XIV.

b) Proof of theorem XVI. We have already proved in a) (utilizing only the boundedness of $F(t)$) that $F(t)$ is *w.a.p.* It is therefore sufficient, making use of the properties of uniformly convex spaces, to prove that R_F is r.c.

We first of all remember that a space X is called a uniformly convex (or Clarkson) space if in the interval $0<\sigma\leq 2$ there exists a function $\omega(\sigma)$, with $0<\omega(\sigma)\leq 1$, such that

$$\|x\|, \|y\| \leqslant 1 \text{ and } \|x-y\| \geqslant \sigma \Rightarrow \left\| \frac{x+y}{2} \right\| \leqslant 1-\omega(\sigma). \tag{5.3}$$

Now we observe that from (5.3) it follows for any x and y:

$$\|x-y\| \geqslant \sigma \max \{ \|x\|, \|y\|\} \Rightarrow \left\|\frac{x+y}{2}\right\| \leqslant (1-\omega(\sigma)) \max \{\|x\|, \|y\|\}. \tag{5.4}$$

Let us assume that the range R_F is not r.c. There exist then a constant $\delta>0$ and a sequence $\{s_n\}$ such that

$$\|F(s_j) - F(s_k)\|\geqslant \delta \qquad (j\neq k). \tag{5.5}$$

We can suppose that $\{s_n\}$ is *regular* with respect to $f(t)$ and $F(t)$, that is

$$\lim_{n\to\infty} f(t+s_n) = f_s(t), \qquad \lim_{n\to\infty}{}^{*}\, F(t+s_n) = F_s(t) \tag{5.6}$$

uniformly. The last relation follows from Bochner's criterion (theorem XIII), noting that the space X is semicomplete (being reflexive).

It also holds that

$$F(t + s_j) = F(s_j) + \int_0^t f(\eta + s_j)\, d\eta$$

and, consequently, for $j\neq k$,

$$\|F(t+s_j) - F(t+s_k)\| \geqslant \|F(s_j) - F(s_k)\| - \left\|\int_0^t (f(\eta+s_j) - f(\eta+s_k))\, d\eta\right\|.$$

If we fix $t\in J$, we will have, by (5.5) and the former part of (5.6),

$$\|F(t+s_j) - F(t+s_k)\| \geqslant \frac{\delta}{2} \text{ for } j > k \geqslant n_t.$$

Therefore, by (5.2),

$$\|F(t+s_j) - F(t+s_k)\| \geqslant \frac{\delta}{2M} \max \{\|F(t+s_j)\|, \|F(t+s_k)\|\}$$

and, by (5.4),

$$\frac{1}{2}\|F(t+s_j) + F(t+s_k)\| \leqslant (1 - \omega(\frac{\delta}{2M})) \max \{\|F(t+s_j)\|, \|F(t+s_k)\|\} \leqslant$$

$$\leqslant (1-\omega\,(\frac{\delta}{2M}))\;M.$$

From the latter part of (5.6) it then follows

$$\|F_s\,(t)\| \leqslant (1\,-\,\omega(\frac{\delta}{2M}))\;M$$

and, consequently,

$$\operatorname*{Sup}_{t\in J}\|F_s\,(t)\| \leqslant (1-\omega(\frac{\delta}{2M}))\;M. \tag{5.7}$$

Relation (5.7) is absurd; from the latter part of (5.6) follows in fact, the weak convergence being uniform,

$$\lim_{n\to\infty}{}^{*}\quad F_s(t-s_n) = F(t)$$

and therefore

$$\|F(t)\| \leqslant \liminf_{n\to\infty}\|F_s(t-s_n)\,\| \leqslant (1-\omega(\frac{\delta}{2M}))\;M,$$

which contradicts (5.2).

It is of interest to note that the previously given example is, in a certain sense, the only possible. Both functions $f(t)$ and $F(t)$ belong in fact to the subspace c_0 of l^∞, of numerical sequences which converge to 0. The analysis of Banach spaces X which do not contain c_0 is due to Pelczynski [10], and the important role of these spaces in the problem of integration was indicated by Kadets [11]. The following theorem in fact holds:

XVII (Kadets) Assume *$f(t)$ a.p., $F(t)$ bounded. Then $F(t)$ is a.p. if and only if the space X does not contain c_0.*

Observation. As we have observed in §1, the above considerations are essential in the study of some typical equations, linear or non linear, of mathematical and theoretical physics; in particular [12]: the wave equation, Schrödinger's equation with time-dependent operator, and, in the non linear field, the wave equation with non linear dissipative term and the Navier-Stokes equation (assuming, in all cases, the presence of an *a.p.* forcing term $f(t)$, and setting the problems in Hilbert or uniformly convex spaces).

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Politecnico di Milano
Dipartimento di Matematica
20133 Milano, Piazza Leonardo da Vinci 32
Italy

Almost Periodicity in Solid State Physics and C*Algebras

By JEAN BELLISSARD

I. *Almost Periodic Physics:*

Several physical phenomena involve almost or quasi periodic functions. The earliest examples concerned applications in Classical Mechanics. More recently almost periodicity has been important in Quantum Mechanics especially in problems involving conductors. Most of the corresponding examples concern Schrödinger operators with quasi or almost periodic potential or some tight binding approximation of it. The aim of this section is to provide physical examples taken from Solid State Physics.

I-1. *Quasi 1D conductors:*

In 1964 Little [Little], in a remarked article suggested that superconductivity could be enhanced in organic conductors. More generally, molecular conductors represent a favourable case for such a mechanism because they may contain easily 20 to 40 time more atoms than a metal in a unit cell and the intermolecular vibrations permit an increase of the interactions between Cooper pairs. These remarks led the community to search for conducting organic crystals. In the early seventies the salts of TTF (tetrathiofulvalene) were produced in particular the TTF-TCNQ. The corresponding molecules are planar and are vertically linked together through hydrogen bridges leading to a very strong anisotropy and also to the existence of a conduction band in the vertical direction. It was soon realized however that most of them even though quite good conductors at room temperature, became insulator at low temperature preventing a superconductor transition to occur. In 1979 Jerome, Bechgaard et al. [Schultz] found a new family of molecules, similar to the TTF salt, the so called TMTSF salts (tetramethyl-tetraselena-fulvalene) providing a superconductor transition at low temperature. Our aim here is not to consider the superconductor transition but rather to provide an explanation for the existence of a metal-insulator transition in the early examples.

In describing the metallic properties of such a chain, one usually ignores the electron interaction, and the only collective constraints comes from Pauli's principle leading to the Fermi-Dirac distribution at thermal equilibrium. It is then sufficient to investigate the one electron Hamiltonian. In our problem since the conductivity is essentially one dimensional, it will be sufficient to represent it as a 1D Schrödinger operator. Thanks to the periodic arrangement of the molecules, the effective potential V seen by a typical conduction electron will be a spatially periodic function of a period "a" determined by the chemical forces. The Bloch theory, the Solid State analog of Floquet's theory, predicts that the energy spectrum is obtained by searching eigenfunctions satisfying Bloch's boundary conditions namely, in suitable units:

$$\{-\frac{\partial^2}{\partial x^2} + V(x)\}\ \psi_k(x) = E(k)\ \psi_k(x) \qquad \psi_k(x+a) = e^{ika}\ \psi_k(x) \tag{1}$$

The electron gas will then occupy all energy levels below the chemical potential which usually coincides at low temperature with the Fermi energy level E_F. However in these systems, because the chemical bonds are not as strong as in metals, the electron gas has another possibility to decrease its overall energy, namely by creating a gap at the energy level (fig. 1). This is called the "Peierls instability" [Peierls (55)]. It is obtained through

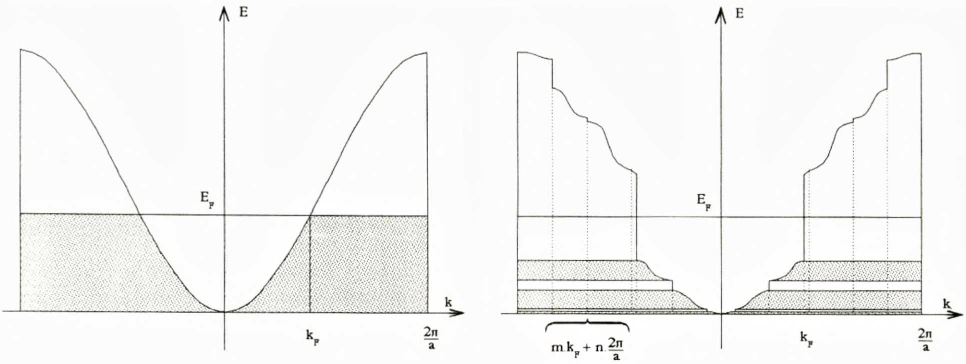

Fig. 1: 1) left: in absence of spatial modulation of the charge density the electron gas occupies the states with energy below the Fermi level.
2) right: if the charge density modulated itself spontaneously, a gap opens at the Fermi level, decreasing the overall energy of the electron gas. This modulation of the CDW is therefore stable (Peierls instability).

a modulation of the electron gas at a spatial frequency $a_F = 2\pi/k_F$ where k_F is the quasi momentum such that $E(k_F) = E_F$. Actually the modulation usually affects the "charge density wave" (CDW), namely the charge distribution in the electron gas along the chain. This effect creates an additional contribution to the effective potential with a spatial period a_F. *Since in general a_F is not commensurate to a the effective potential seen by the conduction electrons is quasi periodic.* Aubry [Aubry 78] proposed, to describe this phenomenon, the following tight binding model, called the *Almost Mathieu equation:*

$$\phi(n+1) + \phi(n-1) + 2\mu\cos 2\pi(x-\alpha n)\ \phi(n) = E\ \phi(n) \tag{2}$$

In this equation, μ represents the strength of the interaction, $\alpha = a_F/a$ is the frequency ratio, and x is a random phase representing the arbitrariness of the origin in the crystal (phason modes). We will see later on in this review that indeed if the extra modulation is strong enough, the corresponding quasi periodic Schödinger operator has a pure point spectrum at low energy leading to exponentially localized states and zero conductivity. It is therefore not surprising to find in general a metal insulator transition at low temperature for these systems. What makes the difference between various molecules is the strength of the Peierls instability. In the TMTSF salts, it seems to be weak enough to avoid the insulator state, and therefore to permit at low temperature the creation of Cooper pairs leading to superconductivity.

I-2. *2D Bloch electrons in a uniform magnetic field:*

The second example of a system described by a quasi periodic potential concerns an electron gas in a two dimensional perfect crystal submitted to a uniform perpendicular magnetic field. This problem has been one of the most challenging encountered by Solid State Physicists. The first proposal to treat it goes back to the thirties with the works of Landau [Landau (30)] and Peierls [Peierls (33)], who gave the lowest order approximation of the effective hamiltonian at respectively high and low magnetic field. The question of finding an accurate effective hamiltonian occupied most of the experts during the fifties (see [Bellissard (88a)] for a short review of that question). The main reason comes from the usefullness of the magnetic field in providing efficient experimental tools for measuring microscopic properties of metals. The Hall effect, the de Haas-van Alfven oscillations, the magnetoresistance, for example provide precise information on the charge carriers, the shape of the Fermi surface, the band spectrum, etc. During the sixties and the seventies, many improvements were obtained on the nature of the corresponding energy spectrum. In particular D. R. Hofstadter computed the spectrum of the so called Harper model as a function of the magnetic flux through a unit cell, end exhibited an amazing fractal structure (see fig. 2) which is still now under

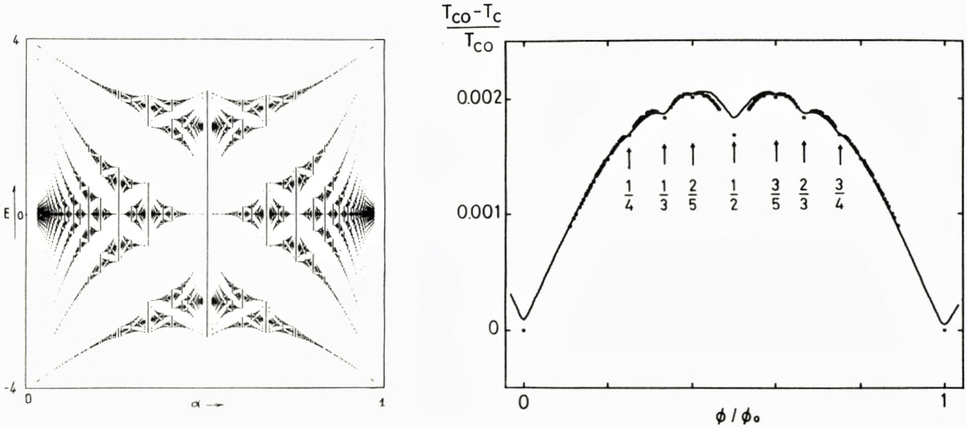

Fig. 2: 1) left: the Hofstadter spectrum as a function of the parameter α.
2) right: measurement of the transition curve between normal and superconduction phase in the (T,B) plane for a square network of filamentary superconductors (taken from [Pannetier (84)]).

study, even though recent results permit to say a lot on it (see [Bellissard (88b)] for a review).

In order to give an idea of how quasi periodicity enters in this game let us consider a rather simple example. Let us assume that an electron be described in a tight binding approximation, by a wave function ψ on a 2D square lattice $\mathbf{Z}^2$.In absence of magnetic field, the energy operator may be effectively described, as a first approximation, by means of nearest neighbours interaction, namely by

$$H\psi\,(m,n) = \psi(m+1,n) + \psi(m-1,n) + \psi(m,n+1) + \psi(m,n-1). \tag{3a}$$

Adding a uniform magnetic field will result in adding a U(1) gauge field, namely in changing the phase of each therm in (2):

$$\begin{aligned} H(B)\;\psi\;(m,n) = \quad & e^{2i\pi A_1(m,n)}\;\psi(m+1,n) + e^{-2i\pi A_1(m-1,n)}\;\psi(m-1,n) \\ & + e^{2i\pi A_2(m,n)}\;\psi(m,n+1) + e^{-2i\pi A_2(m,n-1)}\;\psi(m,n-1) \end{aligned} \tag{3b}$$

where $A_\mu(m,n)$ represents the product of e/h (h being the Planck constant) by the line integral of the vector potential between the point (m,n) of the lattice and the point $(m+1,n)$ for $\mu=1$, or $(m,n+1)$ if $\mu=2$. In particular, because the magnetic field is uniform, one must have:

$$A_1(m,n) + A_2(m+1,n) - A_I(m,n+1) - A_2(m,n) = \frac{\phi}{\phi_0} = \alpha \tag{4}$$

where $\phi_0 = h/e$ is the quantum of flux and ϕ the flux through the unit cell. One solution of the previous equation (4) is the "Landau gauge" namely:

$$A_1(m,n) = 0 \qquad\qquad A_2(m,n) = \alpha m \tag{5}$$

In this case, the operator $H(B)$ commutes with space translations along the n-direction. Therefore the solutions of the stationary Schrödinger equation will have the form:

$$H(B)\;\psi = E\;\psi \qquad \text{with} \qquad \psi(m,n) = e^{-2i\pi kn}\;\phi(n) \tag{6}$$

leading to Harper's equation:

$$\phi(n+1) + \phi(n-1) + 2\cos 2\pi(k-\alpha n)\;\phi(n) = E\;\phi(n) \tag{7}$$

Thanks to eq. (4) "α" is a physical parameter liable to vary, and will be therefore irrational most of the time. The Harper equation appears as a discrete version of a 1D Schrödinger operator with a quasi periodic potential.

If the crystal is not a square lattice but a rectangular one, leading to anisotropy between the two components, the same argument leads to the Almost Mathieu equation:

$$\phi(n+1) + \phi(n-1) + 2\mu\cos 2\pi(k-\alpha n)\ \phi(n) = E\ \phi(n) \tag{7}$$

where μ represents the anisotropy ratio of the coupling constants in the vertical versus the horizontal directions. This equation also represents the effective hamiltonian for a tight binding representation of the effect of a Charge Density Wave in a 1D conductor provided μ represents the strength of the Peierls instability (see eq. (2)).

It is important to remark that (3b), (6) or (7) can be written in an algebraic way by introducing the following two unitaries U and V:

$$U\ \psi\ (m,n) = e^{-2i\pi A_1(m-1,n)}\ \psi(m-1,n) \qquad V\ \psi\ (m,n) = e^{-2i\pi A_2(m,n-1)}\ \psi\ (m,n-1) \tag{8}$$

They satisfy the following commutation relation:

$$U\ V = e^{2i\pi\alpha}\ V\ U \tag{9}$$

The Almost Mathieu hamiltonian can be written as:

$$H = U + U^* + \mu(V + V^*) \tag{10}$$

and in general it is possible to show (see §III.1) that the band hamiltonian for a 2D Bloch electron in a uniform magnetic field belongs to the C*Algebra generated by U and V.

I-3. *Superconductor networks:*

In the Landau-Ginzburg approach [Landau (50)] of the superconductivity, the state of the electron gas is represented by a unique coherent wave function $\Psi(x)$. It plays the role of an order parameter like the magnetization in magnetic systems. The square $|\Psi(x)|^2$ of this wave function will represent phenomenologically the probability density of Cooper pairs in a sort of Hartree approximation. Landau and Ginzburg postulated that the corresponding free energy is given by:

$$F = \int_\Sigma d^3\mathbf{x}\ \{|(\frac{h}{2\mathrm{i}\pi}\partial - 2e\mathbf{A}(\mathbf{x}))\ \Psi(\mathbf{x})\ |^2 + \alpha\,|\Psi(\mathbf{x})|^2 + \beta\,|\Psi(\mathbf{x})|^4 + \frac{|\mathbf{H}(\mathbf{x})|^2}{8\pi}\} \tag{11}$$

where Σ is the volume occupied by the superconductor, e the charge of the electron, ∂ the gradient operator, **A** the vector potential, **H** the effective magnetic field in the bulk, and α, β are temperature dependent phenomenological parameters. To insure the stability of the system, we must have $\beta > 0$. The actual state of the system is provided by functions minimizing the free energy. Since at temperature bigger than the critical temperature there is no Cooper pairs, one must assume that the minimum is reached for $\Psi=0$. This implies in turn that α is positive for $T > T_c$. If $T < T_c$, we must have a non zero solution, and therefore $\alpha < 0$. Assuming a smooth dependence in the temperature, we get:

$$\alpha(T) \approx (T - T_c)\left(\frac{d\alpha}{dT}\right)_{T_c} \qquad \beta(T) \approx \beta_c \qquad \text{at} \quad T \approx T_c \tag{12}$$

In a large superconductor, the magnetic field does not penetrate in the bulk (Meissner effect), unless under the form of quantized flux tubes [Mermin]. The penetration length $\xi(T)$ can be computed in terms of the parameters α and β and is of order of about 1000Å at small T's. This can be seen by computing the minimizing solution of (11) for a half space for instance [Landau (50), Jones]. Near the critical temperature however the penetration length diverges like $\xi(T) \approx \xi_0 (1-T/T_c)^{-1/2}$, and Ψ must be very small, in such a way that the quartic term in (11) may be neglected. Therefore whenever the external magnetic field is uniform, for superconductors of small size, the minimizing solution of (11) is such that $\mathbf{H} \approx$ const. in the bulk and Ψ satisfies the linearized equation:

$$\left\{\frac{h}{2\mathbf{i}\pi}\partial - 2e\mathbf{A}(\mathbf{x})\right\}^2 \Psi(\mathbf{x}) = E\Psi(\mathbf{x}) \qquad E = \left(\frac{d\alpha}{dT}\right)_{T_c}(T_c - T) \tag{13}$$

with some proper boundary condition. To get the minimum of the free energy, *E must be the groundstate of* (13).

These remarks were the basic elements for the study of filamentary superconductors as initiated by DeGennes [deGennes (81)] and Alexander [Alexander], in the study of random mixtures of superconductors and normal metals. The solution of (13) for a thin filament of finite length can be obtained through the one dimensional analog of (13) and a gauge transformation. It is therefore sufficient to know the wave function at the filament ends to know the solution everywhere. The compatibility conditions (current conservation) at the filaments edges give rise to a sort of tight binding representation of the linearized Landau-Ginzburg equation (13). For regular lattices of filamentary superconductors these equations have been written by Alexander, Rammal, Lubensky and Toulouse [Rammal (83)]. For a square lattice of infinitely thin filaments of length "a" one gets:

$$H(B)\ \psi = \varepsilon\ \psi \qquad \varepsilon = 2\cos(a.(E)^{1/2}) = 2\cos\ (a/\xi(T)) \tag{14}$$

where ψ represents the sequence of values of Ψ at the vertices of the lattice, $H(B)$ is the operator given by eq. (3) provided the electron charge e be replaced by the charge of a Cooper pair $2e$ and ε be the groundstate energy of $H(B)$. For real filaments, the thickness is usually not small enough, and a correction due to the bulk must be introduced to fit the experiments.

Eventually the Grenoble group (Chaussy, Pannetier, Rammal and coworkers) performed an experiment on a hexagonal lattice [Pannetier (83)] and a square lattice [Pannetier (84)]: they measured the field dependence of the critical temperature, which is related through (13) to the corresponding groundstate energy of the linearized Landau-Ginzburg equation. The calculation of ε is quite easy numerically and the comparison with the experiment is amazingly accurate (fig. 2). Not only do we get a flux quantization at integer multiples of ϕ_0 ($\phi_0 = h/2e$) but also at fractional values, exactly like in the Hofstadter spectrum. Later on the experiment has been performed on a Penrose lattice, a quasi periodic one [Behrooz], and also on a Sierpinsky gasket [Ghez].

More recently, the Grenoble group realized that the measurement of the magnetic susceptibility near the critical line is related to the derivate of ε with respect to flux ϕ/ϕ_0 thanks to the Abrikosov theory of type II superconductors [Abrikosov]. The Wilkinson-Rammal formula (see [Bellissard (88b)] and section III below) permits to compute this derivative at each rational value of ϕ/ϕ_0. Again the comparison with the experiment is amazingly accurate [Gandit]. The magnetic susceptibility admits a discontinuity at each rational value of ϕ/ϕ_0 in agreement with the Wilkinson-Rammal formula. To date this is the only experiment where these quantities about the Hofstadter spectrum, can be measured so accurately.

I-4. *Normal Conductor networks:*

In a normal metal, one usually explains the weak localization by the existence of an interference increasing the backscattering [Bergmann]. More precisely, due to the slight disorder in the metal, one considers the electron wave as scattered by the randomly distributed impurities. In this process, considering a diffusion path O, A_1, A_2,..., A_n, O' the averaging over the positions A_1, A_2,..., of the scatterers usually decreases the sum of the diffusion amplitude distroying all interference. However, if $O = O'$ (backscattering), the waves following the path forward O, A_1, A_2,... A_n, O and backward O, A_n, A_{n-1},..., A_1, O have no phase difference and they always interfere whatever the position of the scatterers. Thus the backscattering amplitude is higher than the forward scattering, decreasing the electric conductivity.

This effect however occurs as long as the time reversal symmetry is not broken. Adding a magnetic field will decrease the backscattering and the magnetoresistance as well. The phase shift between the two forward and backward paths will be given by

$2\pi e\phi/h$ for each of these paths, namely $2\pi 2e\phi/h = 2\pi\phi/\phi_0$ with now $\phi_0 = h/2e$. Thus as for superconducting systems the effective charge is $2e$ instead but the mechanism is completely different.

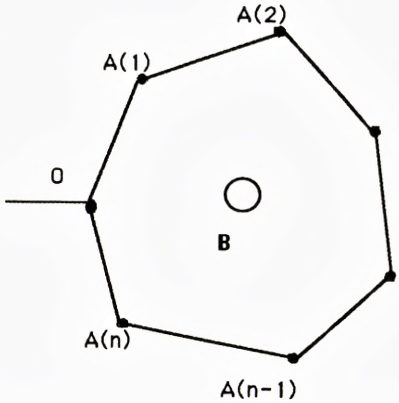

Fig. 3: A typical diffusion path for a quantum wave. The phase shift between the path O, A(1),..., A(n), O and the path O, A(n),..., A(1), O is $2\pi\phi/\phi_0$ where ϕ_0 = h/2e. The replacement of e by 2e comes from the weak localization effect and not from the existence of pairs as in the theory of superconductivity.

The computation of the conductivity is always tricky, in order to take into account the collision time and the phase coherence time. But this weak localization approach gives rise to a correction $\delta\sigma$ to the conductivity given by [Bergmann, Douçot (85)& (86)]:

$$\delta\sigma(x) = -2/\pi\, e^2/h\; C(x,x) \tag{15}$$

where $C(x,x')$ is the Green function defined as the solution of:

$$\{\, (-\mathbf{i}\partial - \frac{2\pi 2e}{h}\mathbf{A}(\mathbf{x}))^2 + \frac{1}{L_\phi^2}\,\}\; C(\mathbf{x},\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}') \tag{16}$$

In this formula, L_ϕ represents the phase coherence length. $C(x,x)$ is usually called the "Cooperon".

One way to investigate this effect consists in looking at a filamentary conductor in which there are loops. If L is the typical loop size, and l the mean free path, one must have $l \ll L$, in order to get weak localization results, but $L \leq L_\phi$ if one wants to observe the interference effect. In this case, the magnetoresistance must be the same for each magnetic field such that the flux through the loop is an integer multiple of ϕ_0. These oscillations have been observed first by Sharvin and Sharvin [Sharvin] on a simple loop and the phenomena are enhanced for a regular lattice of thin wires. Treating eq. 16 as the Landau-Ginzburg equation for a lattice of filamentary superconductors, we eventually obtain the Cooperon from the value of the Green function of the Harper hamiltonian $H(B)$. The actual formula for the resistance of a regular 2D lattice was computed by Douçot and Rammal [Douçot (85) & (86a)] and is given by:

$$\frac{\Delta R}{R} = \kappa \left\{ \frac{\eta ch\eta - sh\eta}{\eta sh\eta} \left(1 - \frac{2}{Z}\right) + 2\, sh\eta\, \tau \left(\frac{\mathbf{1}}{4ch\eta - H(B)}\right) \right\} \tag{17}$$

$$\eta = \frac{a}{L_\phi} \qquad \kappa = \frac{2e^2 L_\phi}{\sigma_0 hS} \tag{18}$$

where Z is the coordination number of the lattice, "a" is the lattice spacing, S is the cross section of the wires, σ_0 is the conductivity of the corresponding perfect conductor as computed by neglecting the weak localization effect, and $H(B)$ is the Harper operator for the corresponding lattice (for a square lattice see eq. 3). In the formula (17), τ represents the trace per unit volume of the operator in parenthesis (see § III).

The measurement of such a resistance has been performed again by the Grenoble group [Douçot (85) (86b)] and the comparison with the experiment is also amazingly good. This is a spectacular confirmation of the validity of weak localization theory.

I-5. *Quasicrystals:*

In 1984, Schechtman, Blech, Gratias, Cahn [Schechtman] found a new kind of crystalline order in an Al-Mn alloy giving rise to a perfect X-ray diffraction pattern with a five-fold symmetry. Since it is well known that no cristalline group in 3D exists with a five-fold symmetry axis [Mermin], they were led to admit that the translation invariance was broken. Nevertheless because of the quality of the diffraction picture, they proposed a quasi periodicity atomic arrangement. In the early seventies, Penrose [Penrose] had produced examples of quasi periodic tillings of the plane, leading to examples with a five-fold symmetry axis. A systematic rigorous framework of his ideas was proposed by de Bruijn [de Bruijn] and new constructions permitted to produce such arrangements in 2D and 3D. One construction consists in projecting a higher dimensional regular lattice onto a 2D or 3D linear subspace with incommensurate slopes. The icosahedral symmetry observed in the original samples, is realized in $\mathbf{Z}^6$, supporting a representation of the icosahedral group [Duneau, Kramer]. This representation can then be decomposed into a direct sum of two irreducible representations of dimension 3 corresponding to subspaces denoted by E_+ and E_-. To get an example of a quasiperiodic lattice the strip method consists in considering the "strip" Σ obtained by translating the unit semi open cube $[0,1)^{x6}$ in $\mathbf{R}^6$ along the E_+ directions, and in projecting all points in $\mathbf{Z}^6 \cap \Sigma$ on E_+ along the E_- direction. If now E_+ is identified with $\mathbf{R}^3$ one gets a sublattice in 3D invariant by the icosahedral group which is obviously quasi periodic by construction. Moreover it can be shown that such a structure is also invariant by a discrete group of dilations generated by some power of the golden mean. This last fact is not so surprising since the golden mean is related to the cosine of $2\pi/5$. If one represents the sites of this lattice by means of the sum of Dirac measures located at each site, the diffraction pattern obtained by taking the Fourier transform of this measure coincides in position and also rather well in intensity with the experimental observation [Gratias]. Other kinds of quasicrystals have been observed with ten-fold, twelve-fold, and more recently eight-fold symmetries [Kuo] giving rise to a new area in crystallography, called "non Haüyan" in contrast to the standard theory originally formulated by Haüy.

Nevertheless we will have eventually to understand the electronic or mechanical properties of such structures. The phonon spectrum, namely the distribution of the vibrational modes is needed to compute the heat capacity of the thermal conductivity of the quasicrystal. The electron spectrum will help in computing the electric conductivity. Unfortunately quasi periodic Schrödinger operators in more than one dimension are not yet understood. This is probably the reason why most of the models investigated up to now are one dimensional. The strip construction in one dimension from $\mathbf{Z}^2$ leads to a chain of points x_n on the real lines such that $x_{n+1}-x_n$ takes on two incommensurate values distributed in a quasiperiodic way. The phonon spectrum for such a crystal can be described by the spectrum of the following discrete Schrödinger equation [Luck]:

$$\psi(n+1) + \psi(n-1) + \lambda\,\chi_A\,(x-n\alpha)\;\psi(n) = E\;\psi(n) \tag{19}$$

where χ represents the characteristic function of the interval A of the unit circle, x is a random phase defined modulo 1 and α is an irrational number. It turns out that the spectral properties of this family of equations are fairly different from the properties of the Harper or Almost Mathieu equations. As was proved by Delyon and Petritis [Delyon (86)], for a large set of α's (19) has no eigenfunctions converging to zero at infinity. Moreover, an argument due to Kadanoff, Kohmoto and Tang [Kadanoff], and Ostlund [Ostlund (83)] supplemented by rigorous proofs of Sütö [Sütö] and Casdagli [Casdagli], shows that for α the golden mean, and λ big enough, the spectrum is a Cantor set of zero Lebesgue measure and non-zero Hausdorff dimension. In particular the spectral measure is singular continuous. The spectrum of (19) as a function of α has been computed numerically by Ostlund and Pandit [Ostlund] and has a simpler structure than the Hofstadter spectrum (fig. 2). This work suggests that the spectrum is a Cantor set of zero Lebesgue measure for any irrational α's. The corresponding eigenstates for α the golden mean were partially computed by Kadanoff, Kohmoto and Tang and also by Ostlund et al. [Ostlund] and exhibit strong recurrence properties in space, being localized around an infinite sequence of points, a result which looks like intermittency. In other words if the wave function is interpreted as the amplitude of the lattice excitation in the crystal, there is an infinite sequence of clusters of atoms far away from each other, in which the lattice oscillations are big whereas the other atoms are essentially at rest.

The corresponding two dimensional model on a Penrose lattice has been studied numerically by Kohmoto and Sutherland [Kohmoto], and is likely to provide also a Cantor spectrum with spatial intermittency. They have discovered also the existence of infinitely degenerate eigenvalues with eigenstates localized in a bounded region (molecular states), like in the case of a Sierpinski gasket [Rammal (84)]. However essentially nothing is known on the nature of the spectrum.

II. *Schrödinger Operators with Almost Periodic Potential:*

In this section we consider a Schrödinger operator H on $\mathbf{R}^D$ (continuous case) or $\mathbf{Z}^D$ (discrete case) with D=1 in most cases and some indications for $D\geq 2$, namely:

$$H\,\psi(x) = -\Delta\psi(x) + V(x)\psi(x) \qquad \psi(x) \in L^2(\mathbf{R}^D)$$

or (1)

$$H\,\psi(x) = -\Sigma_{|e|=1}\,\psi(x-e) + V(x)\psi(x) \qquad \psi(x) \in l^2(\mathbf{Z}^D)$$

where V is almost periodic on $\mathbf{R}^D$ or on $\mathbf{Z}^D$.

These operators exhibit three kinds of properties:

- they tend to have nowhere dense spectra. But it is only a generic property in general; counter examples are known.
- if V is sufficiently smooth, they have a tendency to exhibit a transition between an absolutely continuous and a pure point spectrum when the coupling constant is increased. This is interpreted physically as a metal-insulator transition. In most cases investigated, the eigenfunctions corresponding to the absolutely continuous component are Bloch waves whereas the eigenstates of the pure point spectrum are exponentially localized.
- if the potential is not smooth, if the frequency module is not diophantine or if the coupling constant is critical, the spectrum has a tendency to be singular continuous.

II-1. *Nowhere dense spectra:*

Historically, one of the first rigorous results concerning the gaps of a Schrödinger operator with a quasi periodic potential was provided by Dubrovin, Matveev and Novikov [Dubrovin]. Investigating quasi periodic solutions of the KdV equation by means of the inverse scattering method, they were able to construct a class of potentials for the 1D case giving rise to a spectrum having finitely many gaps. This result has been recently extended by a work of Johnson and de Concini [Johnson] for the case of infinitely many gaps having some regularity property. This family of potentials is obtained by constructing a Jacobi surface depending on the spectrum and a canonical torus associated to that surface, in such a way that V is the restriction of an algebraic function on this torus to the orbit of a constant vector field on this torus. This is the reason why such a potential is called "algebraic-geometric". They constitute a family with a finite number of parameters and for this reason it is non generic in the space of almost or even quasi periodic functions with the same frequency module.

THEOREM 1: The set of almost periodic finite zone potentials with a spectrum given by $\Gamma = \bigcup_{i\in[0,N]}[E_{2i}, E_{2i+1}]$ with $E_{2N+1}=\infty$ is isomorphic to the Jacobian variety $J(\Gamma)$ (namely a 2N-torus) of the Riemann surface $R(\Gamma) = \{(W,E)\in\mathbf{C}^2\,;\, W^2-P_{2N+1}(E)=0\}$ if $P_{2n+1}(E)$ is the polynomial $\Pi_{i\in[0,N]}(E-E_i)$.

◊

In 1980, J. Moser [Moser], J. Avron and B. Simon [Avron (81)] and Chulaevski [Chulaevski] proved a result concerning the generic character of nowhere dense spectra. A limit periodic function f is a continuous function on $\mathbf{R}$ which is a uniform limit of a sequence $\{f_n\}$ of continuous periodic functions on $\mathbf{R}$. If T_n is the period of f_n we must have $T_{n+1}/T_n \in \mathbf{N}$. The same definition applies for limit periodic sequences. Let L be any separable Fréchet topological vector space of limit periodic functions or sequences, we get:

THEOREM 2: If $D=1$, there is a dense G_δ set $L°$ in L such that if $V\in L°$, the operator H in (1) has a nowhere dense spectrum.

◊

An interesting class of the limit periodic models giving rise to a nowhere dense spectrum, is given by Jacobi matrices of a Julia sets. The first example was provided by Bellissard, Bessis and Moussa [Bellissard (82d)], and concerned polynomials of degree 2. Their work was extended to polynomials of higher degree by Barnsley, Geronimo and Harrington [Barnsley (83,85)]. Let P be a polynomial of degree N with real coefficients. One will assume that P is monic, namely $P(z)=z^N+O(z^{N-1})$. One considers the dynamic on the complex plane $\mathbf{C}$ defined by:

$$z(n+1) = P(z(n)) \tag{2}$$

In general it has finitely many attracting fixpoints including the point at infinity, each having an open basin of attraction. The Julia set $J(P)$ of P is the complement of the union of them. It is a compact set. The Fatou-Julia theorem [Fatou, Julia] gives necessary and sufficient condition in order that $J(P)$ be contained in the real line and completely disconnected. When this happens, there is a unique probability measure μ_p on $J(P)$ called the balanced measure, which is both P and P^{-1} invariant. It is singular continuous. The general theory of orthogonal polynomials allows to associate canonically to μ_p a Jacobi matrix (namely an infinite tridiagonal matrix indexed by **N**) in the following way: consider in $L^2(J(P), \mu_p)$ the orthogonal basis p_n obtained from the set of monomial functions $x \in J(P) -> x^n$, $n \in \mathbf{N}$, by the Gram-Schmidt process. It is easy to show that p_n is a monic polynomial of degree n, such that $p_n(P(x)) = p_{nN}(x)$ and $p_0 = 1$. The Jacobi matrix $H(P)$ associated to P is the matrix operator of multiplication by x in $L^2(J(P), \mu_p)$, in the previous basis properly normalized. Since $J(P)$ is compact it follows that $H(P)$ is a bounded operator. By construction, $J(P)$ is the spectrum of $H(P)$ and μ_p is equivalent to its spectral measure. Therefore we get a class of self-adjoint operators having a singular continuous spectrum. The remarkable property of this class lies in the following remark. Let D be the operator on $L^2(J(P), \mu_p)$ defined by $D^*f(x) = f(P(x))$. Due to the invariance properties of the balanced measure, it is easy to see that D is a partial isometry such that [Bellissard (85b)]:

$$DD^* = \mathbf{1} \quad D^*D = \Pi \qquad D(z\mathbf{1} - H(P))^{-1} D^* = P'(z)/N\, \{P(z)\mathbf{1} - H(P)\}^{-1} \tag{3}$$

where Π is the projection onto the subspace generated by the polynomials of the form p_{nN}, $n \in \mathbf{N}$. If one identifies $L^2(J(P), \mu_p)$ with $l^2(\mathbf{N})$ through the basis given by the p_n's, D appears as the dilation operator $Df(n) = f(Nn)$. The main expected result can be summarized in the following conjecture (this part has been only partially solved in [Bellissard (85b)]):

Conjecture: If P is a monic polynomial of degree N with real coefficients, such that no critical points lie in its Julia set, its Jacobi matrix $H(P)$ is the norm limit of a sequence $H_n(P)$ of periodic Jacobi matrices indexed by **N**, with periods N^n.

◊

If P is a polynomial such that the conclusion of the previous conjecture is true, we will say that it has the property *LP*. *LP* has been rigorously proven in the following cases [Barnsley (85)]:

Theorem 3: P has the property *LP* in the following cases:
(i) if $P(z) = z^2 - \lambda$ with $\lambda > 3$.
(ii) if $P(z) = a^N T_N(z/a)$ where T_N is the N^{th} Tchebyshev polynomial and $a > \sqrt{3/2}$.
(iii) if $P(z) = a^N T_N(z/a) + b$ where $N=3$ provided $a \geq 5$, $|b| \leq 5$ or $N=4$ and $a \geq 2$, $|b| \leq 22$.

◊

Another class of limit periodic operator of interest is given by the so-called "hierarchical models". The first examples were provided by Jona-Lasinio, Martinelli and Scoppola [Jona (84&85)], to illustrate ideas on long range tunneling effect. They got a large class of models with nowhere dense spectra and singular continuous spectral measure. Along the same line Livi, Maritan and Ruffo [Livi] introduced a more specific example given by (1) with:

$$V(0)=0 \qquad V(2^n(2l+1))=v(n) \tag{4}$$

for which one can prove rigorously that the spectrum is nowhere dense with zero Lebesgue measure provided $\limsup_{n->\infty}(v(n+1)-v(n))/(v(n)-v(n-1)) > 2$ [Bellissard (87)].

The most challenging problem is obviously the spectrum of the Almost Mathieu operator $H(\alpha,\mu,x)$ defined on $l^2(\mathbf{Z})$ by (1) with $V(n) = 2\mu\cos 2\pi(x-n\alpha)$. Here μ represents a coupling constant and can be restricted to $\mathbf{R}_+$ without loss of generality; a is a real number but since $H(\alpha+1,\mu,x)=H(\alpha,\mu,x)$ it can be seen as an element of the torus $\mathbf{T}$; x is in $\mathbf{T}$ and represents a generic translation on $\mathbf{Z}$ for it is shifted by α when $H(\alpha,\mu,x)$ is translated by 1 on $\mathbf{Z}$. When α is irrational $H(\alpha,\mu,x)$ is periodic and the usual Bloch or Floquet theory applies. Let $\Sigma(\alpha,\mu)$ be the union over x in $\mathbf{T}$ of the spectra of $H(\alpha,\mu,x)$. We first get:

Theorem 4: (i) If α is irrational, the spectrum of $H(\alpha,\mu,x)$ coincides with $\Sigma(\alpha,\mu)$.
(ii) *Aubry-André's duality:* for every α in $\mathbf{T}$, $\mu\Sigma(\alpha,1/\mu)=\Sigma(\alpha,\mu)$.
(iii) *Aubry-André-Thouless's bound:* the Lebesgue measure of $\Sigma(\alpha,\mu)$ is bounded below by $4|1-\mu|$.

◊

(i) Results from the remark that $H(\alpha+1,\mu,x)$ is unitarily equivalent to $H(\alpha,\mu,x)$ by translation, and is is norm continuous with respect to x. Thus its spectrum is unchanged under the shift $x->x+\alpha$, and is continuous with respect to x. The Aubry-André duality is an argument due to Derrida and Sarma [Derrida] and used by Aubry-André [Aubry (78&80)] to exhibit a metal insulator transition. At last Aubry and André discovered numerically the bound on the Lebesgue measure of $\Sigma(\alpha,\mu)$ and Thouless proved it rigorously [Thouless (83)].

In their original work Aubry and André found also that $\Sigma(\alpha,\mu)$ was a Cantor set whenever α is irrational. This was an extension of the work by Hofstadter on Harper's equation [Harper, Hofstadter] (see fig.2). The earliest rigorous result in this context was given by Bellissard and Simon [Bellissard (82c)]:

Theorem 5: There is a dense G_δ set Ω in $[0,1]\,x\mathbf{R}$ such that if $(\alpha,\mu)\in\Omega$ then $\Sigma(\alpha,\mu)$ is nowhere dense.

◊

Actually, one may conjecture that for $\mu \neq 0$ and α irrational the spectrum is nowhere dense. In this result only a generic set of values of α gives this property. This is insufficient for we do not even know whether Ω has positive Lebesgue measure. This theorem has been supplemented by the following result of Sinai [Sinai]:

THEOREM 6: Let α be an irrational number with continued fraction expansion $[a_0, a_1, \ldots, a_n, \ldots]$ such that $a_n \leq \text{const.}\, n^2$. There is $\mu_0 > 0$ such that if $|\mu| > \mu_0$ or if $|\mu| \leq \mu_0$ then the Almost Mathieu operator $H(\alpha, \mu, x)$ has a nowhere dense spectrum of positive Lebesgue measure.

◊

The previous result is in a sense complementary to theorem 5, for the set of α for which theorem 6 holds is the complement of a dense G_δ set but has a full Lebesgue measure.

Another recent result has been provided by Helffer and Sjöstrand [Helffer (87)] using a semiclassical analysis following a renormalization group argument of M. Wilkinson [Wilkinson (84b)]. It concerns specifically the case $\mu=1$, namely the Harper equation.

THEOREM 7: Let E_0 be positive. There is N_0 a positive integer such that for any irrational number α with continued fraction expansion $[a_0, a_1, \ldots, a_n, \ldots]$ such that $a_n \geq N_0$ the spectrum of the Almost Mathieu operator $H(\alpha, \mu=1, x)$ has the following structure:

(i) its convex hull is an interval of the form $[-2+O\,(1/a_1), 2-O\,(1/a_1)]$.

(ii) there is an interval J_0 of length $2E_0 + O\,(1/a_1)$centered at an energy order$O\,(1/a_1)$ such that $SpH(\alpha, \mu, x) \setminus J_0$ is contained in the union of intervals J_i $(N_- \leq i \leq N_+, i \neq 0)$of length $\exp(-C(i)/a_1)$ with $C(i) \approx 1$, separated from each other by a distance of order $O(1/a_1)$.

(iii) for $i \neq 0$ let f_i be the affine increasing map transforming J_i into $[-2,2]$, then $f_i(SpH(\alpha, \mu, x) \cap J_i)$ is contained in the union of intervals $J_{i,k}$ having the same properties as the J_i's provided a_1 be replaced by a_2, and so on.

◊

In this result at each step one has to exclude a central band J_0 , $J_{i,0}, \ldots$, in such a way that nothing can be said on the Hausdorff measure of the spectrum which is believed to be 1/2 from numerical calculations [Tang]. On the other hand, the restriction on α is drastic for if $N_0 \neq 1$, it excludes a set of Lebesgue measure one. However it takes into account the self-reproducing properties seen on the Hofstadter spectrum [Hofstadter].

A complementary result on a wider family of quasi periodic operators will be given in section III (Theorems 6&7).

Another interesting class of almost periodic Schrödinger operator with nowhere dense spectrum, is provided by 1D quasicrystals. The first results were provided simultaneously by Kadanoff et al. [Kadanoff] and by Ostlund et al. [Ostlund]. They considered the following model on $l^2(\mathbf{Z})$:

$$H\psi(n) = \psi(n+1) + \psi(n-1) + \mu\chi_{[-\sigma^3,\sigma^2]}(n\sigma)\ \psi(n) \qquad \sigma = \frac{\sqrt{5}-1}{2} \tag{5}$$

Writing the eigenvalue equation $H\psi(n) = E\psi(n)$ in the form $\Psi(n+1) = M(n)\Psi(n)$ where $\Psi(n)$ is the vector in $\mathbf{C}^2$ with components $(\psi(n), \psi(n-1))$ and $M(n)$ is a $2x2$ matrix depending upon E, they showed that if F_n is the n^{th} Fibonacci number defined by $F_0 = F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$, one obtains $A(n) = M(F_n)M(F_n - 1)\ldots M(1)$ through the following recursion:

$$A(n+1) = A(n-1)A(n) \tag{6}$$

If now $t(n) = trA(n)$, one easily gets:

$$t(n+2) = t(n+1)t(n) - t(n-1) \tag{7}$$

If $T(n) = (t(n-1),\ t(n),\ t(n+1)) \in \mathbf{R}^3$, (6) is equivalent to $T(n+1) = G(T(n))$, where $G(x,y,z) = (y, z, yz - x)$. A constant of the motion is provided by $I(x,y,z) = x^2 + y^2 + z^2 - xyz$ which defines a hypersurface $\Sigma(E)$ in $\mathbf{R}^3$ depending on the choice of μ and E. By looking at those values of E for which the sequence $t(n)$ is bounded one gets a closed subset of the spectrum of H [Kadanoff, Ostlund (83)]. That it is the full spectrum is a result of Sütö [Sütö]. Remarking that G admits some homoclinic point on $\Sigma(E)$ [Kadanoff], Casdagli [Casdagli] described the spectrum by mean of a Markov partition and a symbolic dynamic to prove that the spectrum is a Cantor set of zero Lebesgue measure and non-zero Hausdorff dimension for $\mu > 8$(this value is probably not the optimal one):

Theorem 8: Let H be given by (5):
(i) The spectrum of H is given by the set of energies E such that the sequence $t(n) = tr\{M(F_n)M(F_n - 1)\ldots M(1)\}$ is bounded.
(ii) The spectrum of H is a Cantor set of zero Lebesgue measure and non-zero Hausdorff dimension for $\mu > 8$.

◊

II-2. *The Metal-Insulator transition:*

In their original work Aubry-André [Aubry (80)] gave an argument on the Almost Mathieu operator to show that a metal insulator transition should occur while the coupling constant varies from $\mu < 1$ to $\mu > 1$. This argument called "Aubry-André's duality" was originally provided by Derrida and Sarma [Derrida] and has been interpreted by Aubry-André in the context of 1D incommensurable chains. Let ψ be a sequence indexed by **Z** and solution of the Almost Mathieu equation:

$$\psi(n+1) + \psi(n-1) + 2\mu \cos 2\pi(x-n\alpha)\,\psi(n) = E\,\psi(n) \tag{8}$$

For μ very small, a perturbation argument suggests for ψ an expansion of the form:

$$\psi(n) = e^{2i\pi kn} \Sigma_{p\epsilon\mathbf{Z}} f(p)\, e^{2i\pi p(x-n\alpha)} \tag{9}$$

Taking this ansatz seriously leads for the $f(p)$'s to the following equation:

$$f(p+1) + f(p-1) + 2/\mu \cos 2\pi(k-p\alpha)\, f(p) = E/\mu\, f(p) \tag{10}$$

We recognize the Almost Mathieu equation after changing μ into $1/\mu$ and rescaling the energy E into E/μ. Suppose that (9) converges say uniformly with respect to x, it follows that the sequence $\{f(p); p\epsilon\mathbf{Z}\}$ is certainly square summable, and that for μ small, the "dual equation" (10) admits E as an eigenvalue. Thus $\mu=1$ is critical and separates a regime where perturbation expansion should in principle be relevant leading to Bloch-like waves, namely extended states, whereas at high coupling the previous "duality" argument gives eigenvalues with localized states. Moreover this argument shows that exponential fall-off of the $f(p)$'s, namely exponential localization, implies analytic dependence of the Bloch waves in the parameter $x-n\alpha$.

To go beyond this heuristic argument, one usually introduces the so-called "Lyapounov exponent" γ representing roughly speaking the rate of exponential increase of a generic solution of (8) at infinity. More precisely let $\Psi=(\psi(0), \psi(1))$ be a vector in $\mathbf{C}^2$, then let ψ be the unique solution of (8) with initial conditions given by Ψ. Then γ is defined by:

$$\gamma(E,\mu,\alpha,x,\Psi) = \limsup_{n\to\infty} \frac{\log\left(|\psi(n+1)|^2 + |\psi(n)|^2\right)}{2n} \tag{11}$$

Proposition 1: Let H be given by (8) with α irrational:
(i) $\gamma(E, \mu, \alpha, x, \Psi)$ is independent of Ψ almost surely (Lebesgue measure).
(ii) $\gamma(E, \mu, \alpha, x)$ is non-negative and independent of x almost surely (Lebesgue measure).
(iii) *Herbert-Jones-Thouless formula:* [Herbert, Thouless (72)]: if $\chi_{[-N,N]}$ is the characteristic function of the interval $[-N,N]$, one has:

$$\gamma(E,\mu,\alpha) = \lim_{N\to\infty} \frac{1}{2N+1} Tr\{\chi_{[-N,N]} \log|E-H|\} \tag{12}$$

(iv) *Aubry-André's duality formula* [Aubry (80)]:

$$\gamma(E,\mu,\alpha) = \log \mu + \gamma(E/\mu, 1/\mu, \alpha) \tag{13}$$

(v) *Aubry-André-Herman's bound* [Aubry (80), Herman]:

$$\gamma(E,\mu,\alpha) \geq \log \mu \tag{14}$$

◊

This set of results can be used for getting an information about the nature of the spectral measure:

THEOREM 9: Let H be the self-adjoint operator given by (8):
(i) *Floquet-Bloch theory:* If α is rational, H has purely absolutely continous spectrum.
(ii) *Pastur-Ishii theorem* [Ishii, Pastur]: If α is irrational, for $\mu>1$, the absolutely continuous spectrum of H is empty.
(iii) *Delyon's theorem* [Delyon (87)]: If α is irrational, for $\mu<1$,the point spectrum of H is empty. If $\mu=1$, the point spectrum if it exists is contained in the set of energies where the Lyapounov exponent vanishes, and the eigenstates are in $l^2(\mathbf{Z})$ but not in $l^1(\mathbf{Z})$.

◊

In the rational case H is periodic and the usual Floquet-Bloch theory applies. In particular the eigensolutions of (8) are Bloch waves of the form given by (9), with an energy $E(k)$ depending analytically on k. In the irrational case, due to the Aubry-André-Herman bound, for $\mu>1$ the Lyapounov exponent is positive, and the Pastur-Ishii theorem, which is valid for any 1D Schrödinger operator with random potential, implies the absence of absolutely continuous spectrum. The Delyon result is specific to the Almost Mathieu model since it uses Aubry-André's duality in an essential way.

The question is now to know whether for $\mu>1$ the spectrum is pure point as predicted by Aubry-André's duality. The answer is actually no in general as it follows from the following result by Avron and Simon [Avron (82)]:

THEOREM 10: Let H be given by (8). There is a dense G_δ set Σ of irrational numbers in [0,1] of zero Lebesgue measure such that if $\alpha\epsilon\Sigma$, and $\mu>1$, H has a purely singular continuous spectrum.

◊

This result is actually a special case of a theorem by Gordon [Gordon] which extends to a wide set of examples. On the other hand, the Σ is contained in the set of "Liouville numbers" namely those irrational numbers for which there is a sequence of rational

p_n/q_n such that $|\alpha - p_n/q_n| \leq 1/q_n^{\,n}$ for all n. These numbers are so rapidly approximated by rationals that the solution of (8) look like Bloch waves on long distances, and never succeed to vanish at infinity.

Conjecture: There is a dense G_δ set Σ of irrational numbers in [0,1] such that if $\alpha \in \Sigma$, and $\mu < 1$, H has a purely singular continuous spectrum.

However one can always argue that Liouville numbers are exceptional since they have Lebesgue measure zero. Almost every number is "diophantine" namely for every $\sigma > 2$, there is $C > 0$ such that $|\alpha - p/q| \geq C/q^\sigma$ for all p/q. Using the Kolmogorov-Arnold-Moser method, Dinaburg and Sinai [Dinaburg] got the existence of some absolutely continuous spectrum with Bloch waves for models given by (1) on $\mathbf{R}$. The adaptation of their technics led Bellissard-Lima-Testard [Bellissard (83a)] to a partial proof of the Aubry-André conjecture, in the sense that only a closed subset of positive Lebesgue measure of the spectrum exhibits a metal insulator transition. This result has been recently supplemented by Fröhlich-Spencer-Wittwer [Fröhlich] and by Sinai [Sinai] which gives:

Theorem 11: Let H be given by (8) and let α satisfy a diophantine condition of the form $|\alpha - p/q| \geq C/q^\sigma$ for all p/q for some $\sigma \geq 3$. Then:

(i) there is $\mu_0 > 0$ such that if $\mu \leq \mu_0$, the absolutely continuous component of the spectrum of H is non empty and is supported by a set of Lebesgue measure bigger than $4 - o(1)$ as $\mu -> 0$.The corresponding eigensolutions have the form (9) with exponentially decaying $f(p)$'s.
(ii) there is $\mu_0 > 0$ such that if $\mu \geq \mu_0$,for almost all x, the spectrum of H is pure point with exponentially localized eigenstates.

◊

The previous result has been extended to various examples on the real line in particular [Dinaburg, Fröhlich]:

Theorem 12: Let H be given by (1) on $\mathbf{R}$.

(i) If $V(x) = \Sigma_{n \in \mathbf{Z}^\nu} v(n)\, exp(\mathbf{i} n.\omega\, x)$ with $\omega \in \mathbf{R}^\nu$ satisfying $|n.\omega| \geq C/|n|^\sigma$ for all $n \in \mathbf{N}^\nu$ and some $C > 0$, $\sigma > \nu$. We suppose $\|V\|_r = \Sigma_{n \in \mathbf{Z}^\nu} |v(n)|\, exp(-r|n|) < \infty$. Then there is E_0 real such that in the interval (E_0, ∞) H admits some absolutely continuous spectrum with eigenfunctions given by Bloch waves of the form $\psi(x) = \exp(\mathbf{i}kx)\ \Sigma_{n \in \mathbf{Z}^\nu} f(n)\, exp(\mathbf{i} n.\omega\, x)$ where the Fourier coefficients $f(n)$ decrease exponentially fast.

(ii) If $V(x) = -\mu\{\cos 2\pi x + \cos 2\pi(\alpha x+\theta)\}$ where α satisfies the diophantine condition $|\alpha - p/q| \geq C/q^3$ for all p/q, μ is large enough, then for almost all q, the spectrum of H in the interval $[-2\mu, -2\mu + O(\sqrt{\mu}\ (1+\alpha^2))]$ is pure point with exponentially localized eigenstates.

◊

It is interesting to note that before these results, nice examples of quasi periodic Schrödinger operators on **Z** have been produced. P. Sarnak [Sarnak] investigated a large class of non self-adjoint operators for which he has been able to compute exactly the spectrum, and found also a transition between pure-point and continuous spectrum. One of the simplest examples of Sarnak operators is given by $H(\mu)\psi(n) = \psi(n+1) + \mu\psi(n)\exp(2\mathbf{i}\pi\alpha n)$. Using a KAM algorithm and an inverse scattering method W. Craig [Craig] produced almost periodic potentials having essentially an arbitrary pure point spectrum. Along the same line Bellissard-Lima-Scoppola [Bellissard (83b)] and Pöschel [Pöschel] exhibited a class of unbounded potentials having dense point spectrum on **R**. This class was derived from the "Maryland model" [Fishman (82)] described by Fishman-Grempel-Prange and which is solvable: it is given by (1) on **Z** with $V(n) = \mu\tan\pi(x-n\alpha)$. It has dense pure point spectrum on **R** if α is diophantine, whereas if α is in some class of Liouville numbers it has singular continuous spectrum [Fishman (83), Simon (84)] (see section II-3 below).

Another question is related to the existence of mobility edges, namely points in the spectrum separating pure point from continuous spectrum. This has been observed numerically by Aubry and André [Aubry (80)], and Bellissard-Formoso-Lima-Testard [Bellissard (82b)] found an almost periodic Schrödinger operator on **R** for which mobility edges do exist. However the corresponding potential is not smooth and the existence of mobility edges for smooth potentials is still an open question.

II-3. *Singular continuous spectra:*

In 1978 Pearson [Pearson] gave an example of Schrödinger operators with a potential vanishing at infinity with purely singular continuous spectrum. For a long time this example was considered as pathological and most of the rigorous results in the literature were concerned with sufficient conditions to avoid singular continuous spectra. In the early eighties, when one started getting results for Schrödinger operators with almost periodic or random potentials, the result of Avron-Simon [Avron (82)] (theorem 10) changed completely the situation and one soon realized that singular continuous spectra were not exceptional, if not the rule for problems related with Solid State Physics. One of the most famous still conjectured example is proved by a 2D Bloch electron in a perfect cubic or hexagonal or triangular crystal submitted to a uniform magnetic field such that the flux through a unit cell is an irrational multiple of the flux quantum: the Hofstadter spectrum is a good example.

The argument of Avron and Simon was based on the remark that 1) in a certain regime, the Lyapounov exponent is positive, which by the Pastur-Ishii theorem prevents absolutely continuous spectrum, and 2) that for a certain class of Liouville numbers the potential is extremely well approximated by periodic potentials (Gordon potentials), which implies by Gordon's theorem [Gordon] the absence of point spectrum. The very same argument applies in various situations. For example in the Maryland model namely the equation (1) on **Z** with $V(n) = \tan\pi(x-n\alpha)$, Simon [Simon (84)] defined the quantity $L(\alpha) = \limsup_{n\to\infty} 1/n \log(|\sin(\pi n\alpha)|)$ and proved that if $L(\alpha)=\infty$, the spectrum is purely singular continuous. Fishman-Grempel-Prange [Fishman (83)] investigated the properties of wave functions and found a scale-invariance, showing that they are almost localized on a very sparse sublattice which recurrently reproduces itself at larger scales. The very same argument works as well for a potential of the form $V(n) = 2\mu\cos 2\pi(\alpha n^2+xn+y)$, for both Herman's bound (proposition 1 *(v)*), if α is irrational and Gordon's theorem, if α belongs to a class of Liouville numbers, apply.

In the section II-1. we also introduced the Jacobi matrix of a Julia set, by construction, its spectral measure class is given by the balanced measure on the Julia set. If it is completely disconnected, then one knows that this measure is singular continuous, thus we get another class of singular continuous spectra. For a polynomial of degree 2, $P(z) = z^2-\lambda$, the corresponding Jacobi matrix is given by [Bellissard (82d)]:

$$\begin{array}{lll} H\psi(n) = r(n+1)\psi(n+1)+r(n)\psi(n-1) & n\in\mathbf{N} & \psi(-1)=0 \\ r(0) = 0 \qquad r(2n)^2+r(2n+1)^2 = \lambda & r(2n-1)r(2n)=r(n) & \end{array} \tag{15}$$

THEOREM 13: Let H be given by (15) on **N**. For $\lambda>2$ the spectrum of H is a Cantor set of zero Lebesgue measure and the spectral measure is purely singular continuous. Any point E in the spectrum can be uniquely labelled by an infinite sequence (or coding) $\underline{\sigma}=(\sigma_0,\sigma_1,\ldots,\sigma_n,\ldots)$ of 0's and 1's, such that $E = \sigma_0(\lambda+\sigma_1(\lambda+\sigma_2\ldots)^{1/2})^{1/2}$. The spectral measure on **R** is the image by this map of the Bernoulli measure on the coding. The corresponding eigensolution of $H\psi = E\psi$ satisfies $y_{\underline{\sigma}}(2^k n) = y_{T^k\underline{\sigma}}(n)$ where $\underline{T\sigma}=(\sigma_1,\sigma_2,\ldots,\sigma_{n+1},\ldots)$ and the Lyapounov exponent vanishes on the spectrum.

◊

This result can be extended to any polynomial. It shows in particular that wave functions are not well localized, in contrast with they result of the Maryland group. Moreover they exhibit some chaotic behaviour in space since their value in the large depends upon a random sequence of 0s and 1s.

In the case of 1D quasicrystals Delyon and Petritis [Delyon (86)] proved the following result:

THEOREM 14: Let H be given by (1) on $\mathbf{Z}$ with $V(n) = \mu\chi_A(x-n\alpha)$ where A is an interval on the circle. For Lebesgue almost all α, and any A, the spectral measure of H is purely continuous for Lebesgue almost all x.

◊

Ostlund and Pandit [Ostlund (84)] computed the spectrum of this operator as a function of a and they found a fractal structure suggesting that the Lebesgue measure of the spectrum may be zero. This is an indication that the spectrum may be singular continuous.

At last hierarchical models of Jona-Lasinio, Martinelli and Scoppola [Jona (85)] also give rise to singular spectra. In the case of $\mathbf{Z}$, the class of models described by Li, Maritan, Ruffo [Livi] gives [Bellissard (87)]:

THEOREM 15: Let H be given by (1) on $\mathbf{Z}$ with $V(0) = 0$ and $V(2^n(2k+1)) = v(n)$ for all $k \in \mathbf{Z}$. If $\limsup_{n-\infty}(v(n+1)-v(n))/(v(n)-v(n-1)) > 2$, H has a purely singular continuous spectrum.

◊

These various results show that singular continuous spectra occur normally in many problems of Solid State physics. However Simon et al. [Simon (85&86)] in an argument used for localization gave a result which shows that such spectra are in a certain sense quite unstable under a random perturbation:

THEOREM 16: Let H be a self-adjoint operator having a spectrum supported by a nowhere dense set C of zero Lebesgue measure. Let ψ be a unit vector cyclic for H. Then for Lebesgue almost all μ, the operator $H(\mu) = H+\mu(\psi,.)\,\psi$ has pure point spectrum and the eigenvalues belong to the gaps of C.

◊

This result has been verified for the Jacobi matrix H of the Julia set of a polynomial P by Barnsley-Geronimo-Harrington [Barnsley (85)].

III. *C*Algebras of Almost-Periodic Operators:*

In 1972, Coburn, Moyer and Singer [Coburn] proposed a generalization of the Index formula for elliptic operators on $\mathbf{R}^n$ with almost periodic coefficients. They introduced the C*Algebra of pseudodifferential operators of zeroth order with almost periodic coefficients, and showed that essentially all steps of the usual Index theorem were still valid provided the usual trace of operators be replaced by the trace per unit volume. This idea was exploited later on by Shubin [Shubin] who realized that the integrated density of states of the physicists had a very simple form in this algebraic set-up. In the late seventies, A. Connes generalized the construction to elliptic operators on a foliated compact manifold differentiating along the leaves of the foliation [Connes (82)]. In many cases this C*Algebra admits a natural trace, but there are foliations for which no trace exists. It turns out that most of the problems in Solid State physics involving disordered media, in the independent electrons approximation have hamiltonian affilated to such a C*Algebra [Bellissard (86)]. This has been used to get generic properties of the energy spectrum, such as a gap labelling theorem [Bellissard (82a), (85a), (86), Johnson (82), Delyon (84)], expressions of physical quantities as integrated density of states, Lyapounov exponents, current correlations, for instance. More recently, the definition of a differential structure which is quite natural physically and mathematically, permited to provide a mathematical framework to give a proof of the Quantum Hall Effect [Bellissard (88a&b)] and a detailed study of the Hofstadter spectrum [Bellissard (88b)].

I-1. *Observables and the non-commutative momentum space:*

To start with, let us consider the Almost Mathieu operator. In the section I, eqs. (7, 9, 10) we wrote it in the form:

$$H = U + U^* + \mu(V + V^*) \tag{1}$$

where U and V were two unitaries such that:

$$UV = e^{2i\pi\alpha}\, VU \tag{2}$$

These two operators generate a C*Algebra $\mathcal{A}(\alpha)$ called the rotation algebra. It has been introduced by M. Rieffel [Rieffel] and constitutes a remarkable object in the sense that it is nontrivially non-commutative whenever α is irrational. Nevertheless it is a simple object.

More generally, let H be the Schrödinger operator $H = -\Delta + V$ where V is almost periodic on $\mathbf{R}^D$. The physical system described by H has no longer any translational symmetry and nevertheless it reproduces almost itself under a large translation. On the other hand the translated hamiltonian H_x is equivalent to H in describing the system. Therefore the whole family $\{H_x;\, x \in \mathbf{R}^D\}$ is a natural set of observables. If we insist in performing algebraic calculations, and we need them in practice, we will consider the

C*Algebra generated by $\{H_x;\, x \in \mathbf{R}^D\}$. Since H_x do not commute with H_y for $x \neq y$, this algebra will not be commutative in general. It turns out that this algebra is usually simple to compute in practice [Bellissard (86)]: since V is almost periodic, there is an abelian compact group Ω call the hull of V, a group homomorphism f from $\mathbf{R}^D$ into Ω with a dense image and a continuous function v on Ω such that $V(x) = v(f(x))$. Ω is entirely defined by V or equivalently by H. The C*Algebra $\mathscr{A}$ is then the crossed product $C(\Omega) \times_f \mathbf{R}^D$ of the algebra $C(\Omega)$ of continuous functions on Ω by the action of $\mathbf{R}^D$ defined by f.

A similar treatment can be performed if H is a tight binding approximation, namely an hamiltonian on a lattice like $\mathbf{Z}^D$. It is then sufficient to replace the translation group $\mathbf{R}^D$ by $\mathbf{Z}^D$.

A. Connes developed an analogy with topology or geometry. By Gelfand's theorem (see [Pedersen] for instance), an abelian C*Algebra is isomorphic to the space of continuous functions on a locally compact Hausdorff space vanishing at infinity. Let us decide, by analogy, to identify a non-commutative C*Algebra with the space of continuous functions on some virtual object which will be called a "non-commutative topological space". In our framework, let us consider the special case for which V is periodic. The same construction as before, with the addition of Bloch theory leads to a C*Algebra isomorphic to the tensor product $C(\mathscr{B}) \otimes \mathscr{K}$ where $\mathscr{K}$ is the algebra of compact operators on a separable Hibert space and represents degrees of degenaracy, and $C(\mathscr{B})$ is the space of continuous functions on the Brillouin zone $\mathscr{B}$ (which is usually isomorphic to a torus). Therefore, the periodic case is just the algebra of "functions" over the Brillouin zone (up to the deneracy described by $\mathscr{K}$) which is the crystal analog of the momentum space. By extension, the quasi periodic algebra will be naturally associated to a "non-commutative Brillouin zone". In order that this analogy be efficient, one has to define on these C*Algebras the tools useful in usual geometry: integration theory, differential structure, etc,...

Integration may be obtained through a trace, namely a positive linear (non necessarily bounded) functional τ on $\mathscr{A}$ such that $\tau(AB) = \tau(BA)$ whenever it is defined. It turns out that natural traces can be defined in our situation by mean of a "trace per unit volume". Namely let $d\omega$ be the normalized Haar measure on Ω, which is invariant and (uniquely) ergodic with respect to the action of the translation group $\mathbf{R}^D$. Let also A be an element of $\mathscr{A}$ given by the kernel $a(\omega, x)$ namely acting on $L^2(\mathbf{R}^D)$ through:

$$A_\omega \psi(\mathbf{x}) = \int_{\mathbf{R}^D} d\mathbf{x}'\, a(\omega - f(\mathbf{x}');\, \mathbf{x}-\mathbf{x}')\, \psi(\mathbf{x}') \tag{3}$$

and a similar definition if $\mathbf{Z}^D$ *replaces* $\mathbf{R}^D$. We recall that elements of $\mathscr{A}$ given by smooth kernels are dense in $\mathscr{A}$. The trace per unit volume is then given by:

$$\tau(A) = \lim_{\Lambda \uparrow \mathbf{R}^D} (1/|\Lambda|)\, Tr\{\chi_\Lambda A_\omega\} \tag{4}$$

Using the definition (3) of A and the Birkhoff ergodic theorem (see [Halmos] for instance) one gets for almost every ω in Ω:

$$\tau(A) = \lim_{\Lambda \uparrow \mathbf{R}^D} (1/|\Lambda|) \int d\mathbf{x}'\, a(\omega - f(\mathbf{x}'); \mathbf{0}) = \int_\Omega d\omega\, a(\omega; \mathbf{0}) \tag{5}$$

Actually, in the present case, since the action defined by f is uniquely ergodic (namely the Haar measure is the unique ergodic f-invariant probability measure on Ω) the convergence in (5) is uniform with respect to $\omega \epsilon \Omega$. One can easily check that (5) defines a faithful trace on $\mathcal{A}$ which is unbounded in the case of $\mathbf{R}^D$ but bounded and normalized in the case of $\mathbf{Z}^D$.

A natural differential structure can be defined if one remembers that our algebra is supposed to represent functions on the Brillouin zone: differentiating with respect to momentum variables is usually represented in Quantum Mechanics by commutators with the position operators. Let $\mathbf{X} = (\mathrm{X}_i)_{i \epsilon \{1,\ldots,D\}}$ be the position operator acting on $L^2(\mathbf{R}^D)$ through:

$$\{X_i \psi\}(\mathbf{x}) = x_i\, \psi(\mathbf{x}) \qquad i=1,\ldots,D \tag{6}$$

We define derivations ∂_i on $\mathcal{A}$ by:

$$\{\partial_i A\}_\omega = 2\mathbf{i}\pi\, [\mathrm{X}_i, A_\omega] \tag{7}$$

or equivalently:

$$\partial_i\, a(\omega, \mathbf{x}) = 2\mathbf{i}\pi\, x_i\, a(\omega, \mathbf{x}) \tag{8}$$

The ∂_i's are linear commuting maps on A, satisfying the fundamental formula of derivations, namely $\partial_i(AB) = (\partial_i A)B + A(\partial_i B)$. Moreover $\tau(\partial_i A) = 0$ whenever the formula makes sense. This allows to get an integration by parts formula $\tau((\partial_i A)B) = -\tau(A\partial_i B)$ showing that usual rules in calculus still hold in this non commutative context.

In [Connes (86)] A. Connes gave also a generalization of line or surface integrals of differential forms. In the commutative context they define a de Rham current. In the non commutative case one may define a closed current as follows: giving $A_0, A_1, \ldots, A_n$ in $\mathcal{A}$, one introduces the formal objects $\mathrm{A}_0 dA_1 \ldots dA_n$ by asking that d satisfy the usual rules for a differential, namely $d(AB) = (dA)B + A(dB)$ and $d^2 = 0$; a linear combination of such objects for a fixed n, is called a form of degree n or n-form; let Ω_n be the space of n-forms, and $\Omega(\mathcal{A})$ be the direct sum of the Ω's. One extends the differential d to $\Omega(\mathcal{A})$ by linearity. A closed current is a linear functional τ on $\Omega(\mathcal{A})$ with complex values such that if $\delta\omega$ denotes the degree of ω:

(i) $\tau(\omega_1\omega_2) = (-)^{\delta\omega_1\delta\omega_2}\tau(\omega_2\omega_1)$
(ii) $\tau(d\omega) = 0$
(iii) τ is a current of degree p whenever $\tau(\omega) = 0$ for every $\omega\in\Omega_n$ if $n\neq p$.

As it has been shown by A. Connes in [Connes (86)], a closed current is entirely defined by the $(n+1)$-linear mappings τ_n: $(A_0, A_1,\ldots,A_n)\in\mathcal{A}\rightarrow\tau(A_0 dA_1\ldots dA_n)$ characterized by the following relations:

(a) $$\tau_n(A_1, A_2,\ldots,A_n, A_0) = (-1)^n\,\tau_n(A_0,\ldots,A_n)$$

(b) $$\sum_{j=0}^{n}(-1)^j\,\tau_n(A_0,\ldots, A_j A_{j+1},\ldots,A_{n+1}) + (-1)^{n+1}\tau_n(A_{n+1}A_0, A_1,\ldots,A_n) = 0$$

This non commutative differential form theory gives rise to some cohomology namely the Connes cyclic cohomology and a generalization of the abstract index theorem which has already been used partially to the mathematical proof of the Quantum Hall Effect [Bellissard (88a)].

Let us also indicate that besides the previous algebras, one has other physically relevant examples of observable algebras even in the quasi periodic context. For if one considers the situation in which a two dimensional Bloch electron is submitted to a uniform magnetic field B perpendicular to the plane where the electron lies, the hamiltonian is now:

$$H = \frac{1}{2m}\sum_{i=1,2}(P_i - eA_i(\mathbf{X}))^2 + V(\mathbf{X}) \tag{9}$$

where $\mathbf{P}=(P_1,P_2)$ is the momentum operator (namely $P_i=h/2\mathbf{i}\pi\,\partial_i$), $\mathbf{A} = (A_1,A_2)$ is the magnetic vector potential solution of $\partial_1 A_2-\partial_2 A_1 = B$, m is the electron effective mass, e its electric charge, and V is a periodic potential. The kinetic part is no longer translation invariant because the vector potential breaks the translation symmetry. However adding a phase factor to the translation operator we get the following "magnetic translation" [Zak] on $L^2(\mathbf{R}^2)$:

$$\{U(\mathbf{a})\,\psi\}(\mathbf{x}) = \mathrm{e}^{i\pi eB\mathbf{x}\wedge\mathbf{a}/h}\,\psi(\mathbf{x}-\mathbf{a}) \tag{10}$$

If $\mathbf{a}$ is a period of V, H commutes with $U(\mathbf{a})$. One can then show that the algebra generated by bounded functions of H and its translated is generated by operators A given by a kernel $a(\omega,\mathbf{x})$ defined on $T^2\times\mathbf{R}^2$ as follows:

$$\{A\psi\}(\mathbf{x}) = \int_{\mathbf{R}^2} d^2\mathbf{x}'\; a(-\mathbf{x},\mathbf{x}'-\mathbf{x})\,e^{\mathbf{i}\pi eB\mathbf{x}\wedge\mathbf{x}'/h}\,\psi(\mathbf{x}') \tag{11}$$

In much the same way one gets a trace per unit volume and a differential structure on it.

A lattice version of this algebra is precisely given by the rotation algebra we defined in the beginning of this section. The trace and the differential structure are entirely defined by the following conditions:

$$\tau\,(U^m V^n) = \delta_{m,0}\,\delta_{n,0} \tag{12 a}$$

$$\partial_1 U = 2\mathbf{i}\pi U \qquad \partial_1 V = 0 = \partial_2 U \qquad \partial_2 V = 2\mathbf{i}\pi V \tag{12 b}$$

We see that U and V become analogous to the coordinate functions $e^{2\mathbf{i}\pi x_i}$ $(i=1,2)$ of a 2-torus, if the trace is replaced by the normalized Haar measure, if the ∂_i's represent the usual partial derivatives. Because of (2) however this torus is non-commutative.

If one considers the problem of an electron on a quasicrystal submitted to a uniform magnetic field ond will get another kind of algebra which will be hopefully described in a further work [Bellissard (88c)].

III-2. *Gap labelling and K-Theory:*

In the section II we saw that a Schrödinger operator with almost periodic potential has a tendency to exhibit a Cantor spectrum. In particular it must have infinitely many gaps in a bounded interval. The question is whether there is a canonical way of labelling the gaps which is stable under perturbations or under modifications of the frequency module. It happens that this question is related to the computation of integers in the Quantum Hall Effect, and this justifies a complete study. The first gap labelling was provided by Claro and Wannier by a heuristic analysis of the Hofstadter spectrum [Claro (78)]. This labelling was stable under changes of the magnetic field eventhough the spectrum itself is modified in an intricate manner. The first rigorous results came in 1981 with the works of Johnson and Moser [Johnson (82)] and the result of Bellissard-Lima-Testard [Bellissard (82a, 85a, 86)]. A proof in the case of the Almost Mathieu equation was provided by Delyon and Souillard [Delyon (84)]. Johnson and Moser proved it for the case of a 1D Schrödinger operator with an almost periodic potential using ODE technics. But BLT used an algebraic approach namely the K-theory of C*Algebras and got general results valid in any dimension and for any reasonable pseudodifferential operator with almost periodic or even random coefficients [Bellissard (86)]. They used many of the powerful results discovered in the early eighties by the experts in C*Algebras and especially several explicit formular due to A. Connes. It is our aim here to summarize these results.

Let $\mathscr{A}$ be one of the C*Algebras of operators built in the previous section. Let also H be a self-adjoint operator on $L^2(\mathbf{R}^D)$ (or on $l^2(\mathbf{Z}^D)$) bounded from below such that bounded continuous functions of H belong to $\mathscr{A}$. Physicists introduce first the integrated density of states (the IDS) in the following way:

$$\mathcal{N}(E) = \lim_{\Lambda \uparrow \mathbf{R}^D} (1/|\Lambda|) \, \# \, \{\text{eigenstates of } H|_\Lambda \text{ with energy } \leq E\} \tag{13}$$

where # denotes "the number of", and $H|_\Lambda$ is the operator obtained by restricting H (say in the sense of forms) to a domain D of functions supported by Λ dense in $L^2(\Lambda)$, whenever this makes sense. One can show that because bounded continuous functions of H belong to $\mathcal{A}$, if $\mathcal{A}$ is one of the previous algebras, $H|_\Lambda$ has discrete spectrum bounded from below and therefore the definition of the IDS makes sense. It turns out that the previous formula can be written in a purely algebraic way thanks to the *Shubin formula* [Bellissard (86), Shubin]: if χ_Σ represents the characteristic function of the set Σ, one gets

$$\mathcal{N}(E) = \tau\,(\chi_{(-\infty,E]}(H)) \tag{14}$$

where τ represents the trace on $\mathcal{A}$ which is extended to the von Neumann algebra generated by $\mathcal{A}$ in the GNS representation of the trace. From this formula it follows that $\mathcal{N}(E)$ is a non decreasing function of E which is constant on the gaps of H. *Thus one way way of labelling the gaps is to affect to it the value of $\mathcal{N}(E)$ for E in this gap.* On the other hand whenever E belongs to a gap of H, $\chi_{(-\infty,E]}(H)$ is actually a continuous and bounded function of H and therefore it belongs to $\mathcal{A}$ and it is also a projection. By Shubin's formula (14) the trace of this projection coincides with the value of the IDS on the gap. This trace actually depends only upon the equivalence class of the projection under unitary transformation. On the other hand $\mathcal{A}$ is a separable C*Algebra and by standard results [Pedersen], the set of such equivalence classes is countable. Therefore the *set of values obtained by taking the traces of projections in $\mathcal{A}$ is a countable subset of the positive real line.* Is it possible to get a rule for its computation?

The answer is actually yes, and the K-theory is the key for it [Atiyah]. For indeed if P is a projection in A, let [P] be its equivalence class as defined by von Neumann, namely the set of projections P' such that there are S and T in $\mathcal{A}$ for which $ST=P$ and $TS=P'$. One can check that if P and Q are orthogonal projections, namely if $PQ=QP=0$, their direct sum $P\oplus Q$ coincides with $P+Q$ and is still a projection in $\mathcal{A}$. Moreover its class $[P\oplus Q]$ depends only upon the classes $[P]$ and $[Q]$ and can be denoted $[P]+[Q]$,defining on the set of classes an addition. This law is not always everywhere defined for it may happen that giving P and Q in $\mathcal{A}$, there is not always a pair $P'\epsilon[P]$ and $Q'\epsilon\,[Q]$ such that $P'Q'=Q'P'=0$. However, if one enlarges the algebra by taking the C*Algebras $\mathcal{A}\oplus\mathcal{K}$ generated by finite rank matrices over $\mathcal{A}$, one can show that this is always possible to define the sum of two arbitrary equivalence classes. Then by a canonical construction due to Grothendieck one extends this set into an abelian group, which is called $K_0(\mathcal{A})$. If $\mathcal{A}$ is separable this group is countable.

The trace $\tau(P)$ of a projection P in $\mathcal{A}$ has the property that it depends only upon the class $[P]$. Moreover, $\tau(P\oplus Q) = \tau(P)+\tau(Q)$. Therefore it extends into a group homomorphism τ^* from $K_0(\mathscr{A})$ into the real line. From which it follows that (i) the set of real numbers given by the traces of projections in $\mathscr{A}$ is a generating subset of the countable subgroup $\tau^*(K_0(\mathscr{A}))$ of **R,** (ii) the gap labelling defined in this way satisfied sum rules since $\tau^*(K_0(\mathscr{A}))$ is a group. Thus (see [Bellissard (86)]):

Theorem 1 (first gap labelling theorem): If E belongs to gap of H, the IDS $\mathscr{N}(E)$ belongs to the countable subgroup of **R** given by the image $\tau^*(K_0(\mathscr{A}))$ of the K_0-group of $\mathscr{A}$ under the trace homomorphism.

◊

This theorem is actually useless as long as we cannot compute explicitly the K-group. This has been done for the first time for the rotation algebra by Pimsner and Voiculescu [Pimsner] using earlier results of M. Rieffel [Rieffel]:

Theorem 2: Is $\mathscr{A}_\alpha$ is the rotation algebra generated by two unitaries U and V such that $UV=e^{2i\pi\alpha}VU$, its K-group is isomophic to $\mathbf{Z}^2$ and the image $\tau^*(K_0(\mathscr{A}_\alpha))$ of its K-group by the trace homomorphism is $\mathbf{Z}+\alpha\mathbf{Z}$. If P is a projection in $\mathscr{A}_\alpha$ there is a unique integer n such that $\tau(P)=\{n\alpha\}$ where $\{x\}$ denotes the fractional part of x.

◊

The last part of this theorem comes from the remark that since the trace on the rotation algebra is normalized the trace of a projection must belong to the inverval [0,1].

Soon after this result appeared, A. Connes gave a general formula for computing the K-group and its image under the trace homomorphism [Connes (82)]. We will not give it here in detail but we will only give the result one gets in the case of quasi periodic pseudodifferential operators [Bellissard (85a)]:

Theorem 3: Let H be a pseudodifferential operator on $L^2(\mathbf{R}^D)$ with quasiperiodic coefficients. Let $\mathbf{T}^\nu$ be the hull of its coefficients, and let $f(\mathbf{x})_\mu=\Sigma_i\alpha_{\mu i}x_i$ be the corresponding homomorphism with dense image from $\mathbf{R}^D$ into $\mathbf{T}^\nu$. If E belongs to a gap of H, the IDS $\mathscr{N}(E)$ belongs to the subgroup L of **R** given by $L=\Sigma_{(k)}\mathbf{Z}\alpha_{(k)}$ where the $\alpha_{(k)}$'s are the minors of maximal rank of the matrix $\alpha_{\mu i}$. If $D=1$, L coincides with the frequency module of the coefficients of H.

◊

Theorem 4: Let H be a finite difference operator on $l^2(\mathbf{Z}^D)$ with quasiperiodic coefficients. Let $\mathbf{T}^\nu$ be the hull of its coefficients, and let $f(\mathbf{n})_\mu=\Sigma_i\alpha_{\mu i}n_i$ be the corresponding homomorphism with dense image from $\mathbf{Z}^D$ into $\mathbf{T}^\nu$. If E belongs to a gap of H, the IDS $\mathscr{N}(E)$ belongs to the subgroup L of **R** given by $L=\Sigma_{(k)}\mathbf{Z}\alpha_{(k)}$ where the $\alpha_{(k)}$'s are the minors of any rank of the matrix $\alpha_{\mu i}$ including 1 as a minor of rank zero.

If $D=1$, L coincides with $\mathbf{Z}+F$ where F is the frequency module of the coefficients of H.

◊

Various applications of these gap labelling theorems can be found in [Bellissard (86)], especially in connection with the non existence of gaps. Indeed H will have a Cantor spectrum only if the subgroup $\tau^*(K_0(\mathcal{A}))$ is dense in **R.** Algebraic arguments show that there are examples for which this cannot happen.

In order to illustrate the power of this approach let us however give one example for which the Johnson-Moser argument could not work but the K-theory predicts the result. Consider the operator H_x on $l^2(\mathbf{Z})$ defined by:

$$H_x\psi(n) = \psi(n+1) + \psi(n-1) + \mu\chi_{(0,\beta]}(x-n\alpha)\psi(n) \tag{15}$$

It was shown in [Bellissard (82e)] that the values of the IDS on certain gaps did not follow the rules given by theorem 4 when α and β were rationally independent. Thanks to a recent result of Putnam, Schmidt and Skau [Putnam (85)&(87)] it is possible to compute the K-group of the C*Algebra generated by the H_x's and one finds:

Proposition 1: Let H_x be given by (15) on $l^2(\mathbf{Z})$ where 1, α and β are rationally independent. If E belongs to a gap of H_x, the IDS $\mathcal{N}(E)$ belongs to the countable subgroup $\mathbf{Z}+\mathbf{Z}\alpha+\mathbf{Z}\beta$ of **R.**

◊

III-3. *Spectrum boundaries:*

As we saw in §I-3, the measurement of the normal metal-superconductor transition curve for a network of superconductors in the temperature magnetic field parameters is equivalent to the measurement of the ground state of the Hofstadter spectrum as a function of the parameter α. This raises the question of computing the spectrum boundaries of a self-adjoint element of the rotation algebra $\mathcal{A}_\alpha$ as a function of α. Unfortunately, the algebras $\mathcal{A}_\alpha$ and $\mathcal{A}_{\alpha'}$ are isomorphic if and only if $\alpha=\pm\alpha'$ (mod. 1). Nevertheless there are many quantities of interest which are obviously continuous functions of this parameter. To overcome this difficulty, one can remark that the family $\{\mathcal{A}_\alpha;\ \alpha\in\mathbf{T}\}$ is a continuous field of C*Algebras in the sense of [Dixmier]. To see this more precisely, let us define the universal rotation algebra $\mathcal{A}$ as the C*Algebra generated by three unitaries U, V, λ, such that:

$$[U,\lambda] = 0 = [V,\lambda] \qquad UV = \lambda VU \tag{16}$$

This algebra is mapped onto $\mathscr{A}_\alpha$ through the *homomorphism ρ_α defined by:

$$\rho_\alpha(U) = U \qquad \rho_\alpha(V) = V \qquad \rho_\alpha(\lambda) = e^{2i\pi\alpha} \tag{17}$$

In much the same way to any closed subset J of $[0,1]$, one associates the algebra $\mathscr{A}(J)$ obtained by restricting the elements of $\mathscr{A}$ to J (the norm on $\mathscr{A}(J)$satisfies $\|A\| = \sup_{\alpha\epsilon J}\|\rho_\alpha(A)\|$). The next theorem of G. Elliott [Elliott] gives continuity properties with respect to α:

Theorem 5: Let H belong to the universal rotation algebra. Then:
(i) If $H=H^*$ the gap boundaries of the spectrum of $\rho_\alpha(H)$ are continuous functions of α.
(ii) The norm $\|\rho_\alpha(H)\|$ is a continuous function of α.

◊

This theorem has been supplemented by a theorem of Avron and Simon [Avron (82)]: the spectrum of a Schrödinger operator describing a particle in a uniform magnetic field is continuous with respect to the magnetic field.

It turns out from numerical calculations that the gap boundaries are usually not smooth functions of α (see fig. 2). This has been recently proved by Bellissard (announced in [Bellissard (88b)] following semiclassical ideas developed by Wilkinson [Wilkinson (84a&b)] and Rammal et al. [Wang (87a&b)]. To see this we need further notations.

Let $\mathscr{H}(\mathbf{k}, \alpha)$ be a continuous function of the variables $\mathbf{k} = (k_1, k_2) \epsilon \mathbf{R}^2$ and $\alpha\epsilon(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$. We assume that it satisfies the following properties:

(i) $\mathscr{H}$ is periodic with respect to $\mathbf{k}$ of period 2π in each component of $\mathbf{k}$.
(ii) If $\mathscr{H} = \Sigma_{m\epsilon\mathbf{Z}^2}\, h(\mathbf{m},\alpha)\, e^{\mathbf{i}\mathbf{k}\wedge\mathbf{m}}$ is its Fourier expansion (where $\mathbf{k}\wedge\mathbf{m} = k_1 m_2 - k_2 m_1$) then either:

$$\|\mathscr{H}\|_{(k)} = \sup_{i\leq k}\Sigma_{m\epsilon\,\mathbf{Z}^2} |\partial^i h(\mathbf{m},\alpha)/\partial\alpha^i|(1+|\mathbf{m}|)^k < \infty \qquad \text{for some } k>2 \tag{17a}$$

or the $h(\mathbf{m},\alpha)$'s are holomorphic in α in a strip of width r and

$$|||\mathscr{H}|||_r = \sup_{|Im(\alpha)|\leq r}\Sigma_{\mathbf{m}\epsilon\,\mathbf{Z}^2} |h(\mathbf{m},\alpha)|\, e^{r|\mathbf{m}|} < \infty \qquad \text{for some } r > 0 \tag{17b}$$

(iii) For each α in $(-\varepsilon, \varepsilon)$ the function $\mathbf{k} \to \mathscr{H}(\mathbf{k},\alpha)$ has a unique regular minimum in each cell of period. Without loss of generality one can assume that this minimum is located at $\mathbf{k}=0$ for $\alpha=0$ and that $\mathscr{H}(\mathbf{0},0) = 0$.

Correspondingly we define the quantization of $\mathscr{H}$ as the following H element of $\mathscr{A}$:

$$\rho_\alpha(H) = \Sigma_{\mathbf{m}\epsilon\mathbf{Z}^2}\, h(\mathbf{m},\alpha)\; W(\mathbf{m}) \qquad W(\mathbf{m}) = e^{\mathbf{i}\pi\alpha m_1 m_2} U^{m_1}\, V^{-m_2} \tag{18}$$

The ground state energy $E(\alpha)$ is defined as the infimum of the spectrum of $\rho_\alpha(H)$ in $\mathcal{A}_\alpha$. Our first result concerns the asymptotic behaviour of the bottom of the spectrum of $\rho_\alpha(H)$ as $\alpha\text{−>}0$ namely:

THEOREM 6: Let $\mathcal{H}$ satisfy (i), (ii), (iii) and let H be given by (18). Then there is $E_c > 0$ and $\varepsilon_c \leq \varepsilon$ depending only on $\mathcal{H}$ such that if $\alpha\epsilon\,(-\varepsilon_c, \varepsilon_c)$ the spectrum of $\rho_\alpha(H)$ below E_c is contained in the union of the intervals $\Sigma_n = [E_n(\alpha) - \delta(\alpha), E_n(\alpha) + \delta(\alpha)]$ where if $\mathcal{H}$ satisfies (17a) and δ is equal to $\min(3,k)$:

$$E_n(\alpha) = (2n+1)2\pi\,|\alpha|\,\det{}^{1/2}\{1/2\; D^2.\mathcal{H}(\mathbf{0},0)\} + \alpha\,\partial.\mathcal{H}/\partial\alpha(\mathbf{0},0) + O\,(|\alpha|^{\delta/2}) \tag{19}$$

$$0 < \delta(\alpha) \leq C_1 \mid \alpha \mid^{\delta/2} \tag{20}$$

Here C_1 is a constant depending on $\mathcal{H}$.
If $\mathcal{H}$ satisfies (3b), the estimate (7) is replaced by:

$$0 < \delta(\alpha) \leq \mathrm{C}_1\, e^{-C_2/\alpha} \tag{21}$$

where $C_2 \leq r$.

◊

The proof of this result is a consequence of the semiclassical analysis by Briet-Combes-Duclos [Briet, Combes] and Helffer-Sjöstrand [Helffer (84)]. We then remark that if now $\alpha{=}p/q\epsilon\mathbf{Q},\rho_\alpha(H)$ can be computed by mean of the Floquet theory, and admits a band spectrum. If α is close to p/q, the algebra $\mathcal{A}_\alpha$ can be seen as the subalgebra of $M_q \otimes \mathcal{A}_{\alpha-p/q}$ generate by the elements:

$$U_\alpha = W_1 \otimes U_{\alpha-p/q} \qquad V_\alpha = W_2 \otimes \mathrm{V}_{\alpha-p/q} \tag{22}$$

where W_1 and W_2 are qxq unitary matrices such that $W_i^q = \mathbf{1}$ and $W_1 W_2 = e^{2\pi p/q}\, W_2 W_1$. This is a kind of Renormalization Group analysis which was suggested by the work of Wilkinson [Wilkinson (84b)]. Then the limit $\alpha\text{−>}p/q$ can be analysed by using the theorem 6 and the functional calculus to reduce $\rho_\alpha(H)$ on one band of the spectrum. One gets the following result:

THEOREM 7: Let $\mathcal{H}$ satisfy (i), (ii), (iii) above with $\partial h(\mathbf{m},\alpha)/\partial\alpha = 0$, $k \geq 3$, and let B be a non degenerate band of H at $\alpha = p/q$. The lower (resp. the upper) edge of the band $E^-(\alpha)$ (resp. $E^+(\alpha)$) is given by the *Wilkinson Rammal formula:*

$$E^{\pm}(\alpha) = E^{\pm}(\frac{p}{q}) - (\pm)\, a\, |\, \alpha - \frac{p}{q}\,| + b\,(\alpha - \frac{p}{q}) + O\,(\,|\,\alpha - \frac{p}{q}\,|^{3/2}) \qquad (23)$$

with:

$$a = \frac{2\pi q^2}{\rho(E^{\pm}(\frac{p}{q}))} \quad ; \quad b = \frac{1}{4\mathbf{i}\pi}\, tr_q \{P(\mathbf{k})(\partial_1 H(\mathbf{k})\, \partial_2 P(\mathbf{k}) - \partial_2 H(\mathrm{k})\, \partial_1\, P(\mathbf{k}))\,\}_{E_B(\mathbf{k})=E^{\pm}(\frac{p}{q})} \qquad (24)$$

◊

The first term in this formula is therefore the value of the energy at the band edge for $\alpha=p/q$. The second represents a harmonic oscillator effect, and it produces a discontinuity in the derivate. It shrinks the spectrum in such a way that the neighbouring gap actually increases in size due to this term. The last term comes from a Berry phase, namely from the fact that the eigenprojection $P(\mathbf{k})$ at the value $\alpha=p/q$ defines in general a non trivial line bundle over the 2-torus [Berry, Simon (83)]. This last term accounts for an asymmetry of the derivate around $\alpha=p/q$ and may partially destroy the effect of the first one on the enlargement of the corresponding gap. The derivative of the magnetization of the superconducting array at the transition with respect to the temperature is actually a simple function of the asymmetry of the derivate at each rational point (see [Wang (87a)].

The previous theorem is established for an element H such that $\partial h(\mathbf{m},\alpha)/\partial\alpha=0$. If there is $\mathbf{m}$ such that $\partial h(\mathbf{m},\alpha)/\partial\alpha\neq 0$ one gets an additional contribution to the second term which we will not give here but which is easy to compute.

One consequence of this formula is the following:

COROLLARY: Let $E(\alpha)$ be a gap boundary for $H\in\mathcal{A}_\alpha$ satisfying (i), (ii), (iii). For any irrational value of α, $E(\alpha)$ is differentiable.

◊

OPEN PROBLEM: Is it possible from this formula to get a proof that the spectrum of H is actually a Cantor set for any irrational α?

Following the strategy of Wilkinson, Helffer-Sjöstrand [Helffer (87)] gave more details in the case of Harper's model, using special positivity properties of its quantization, and their result is the content of the theorem 7 in §II-1.

To finish this section let us indicate that in the previous theorem 7 we used a new type of differential calculus similar to the Ito calculus in stochastic differential equations [Bellissard (88b)]. Namely let A be a polynomial in U, V, λ. One can expand A as follows:

$$\rho_\alpha(A) = \sum_{\mathrm{m}\in Z^2} \mathbf{a}(\mathbf{m};\alpha)\; W(\mathbf{m}) \qquad \text{with} \qquad W(m_1,m_2) = e^{\mathbf{i}\pi\alpha m_1 m_2} U^{m_1} V^{-m_2} \qquad (25)$$

We define the operation ∂ by the following formula:

$$\rho_\alpha(\partial A) = \sum_{\mathbf{m}\in\mathbf{Z}^2} \frac{\partial a(\mathbf{m}:\alpha)}{\partial\alpha} W(\mathbf{m}) \tag{26}$$

$C^1(\mathcal{A})$ will denote the completion of the set of polynomials under the norm:

$$\|A\|_{C^1} = \|\partial A\| + \|\partial_1 A\| + \|\partial_2 A\| + \|A\| \tag{27}$$

This operation satisfies the following rules:

THEOREM 8: 1) If A and B belong to $C^1(\mathcal{A})$:

$$\partial(AB) = \partial A\ B + A\partial B + \mathbf{i}/4\pi\ \{\partial_1 A \partial_2 B - \partial_2 A \partial_1 B\} \tag{28}$$

2) If A belongs to $C^1(\mathcal{A})$ and if it is invertible in $\mathcal{A}$, its inverse belongs to $C^1(\mathcal{A})$ and

$$\partial(A^{-1}) = -A^{-1}\{\partial A + \mathbf{i}/4\pi(\partial_1 A\ A^{-1}\partial_2 A - \partial_2 A\ A^{-1}\partial_1 A)\}\ A^{-1} \tag{29}$$

◊

The formula (29) is actually the key point in proving (23) & (24) for α close to p/q once they are proved for α small.

Centre de Physique Théorique and Université de Provence,
CNRS, Luminy, Case 907, 13288, Marseille, Cedex 09, France.

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Notes added in proof:

Since the manuscript was written several problems mentioned in the text have been solved:

I. Let us consider the model described in §I-5 (eq.19) with $A = (1-\alpha, 1]$ and also in §II-1 (eq.5), §II-2 (theorem 14) and given by the following hamiltonian on $l^2(\mathbf{Z}^2)$:

$$H(x,\alpha)\ \psi(n) = \psi(n+1) + \psi(n-1) + \lambda\ \chi_{(1-\alpha,1]}(x\text{-}n\alpha)\ \psi(n).$$

Its spectrum has been computed numerically by:

[128] Ostlundt, S., Kim, S. H.: Renormalization of Quasiperiodic Mappings, *Physica Scripta*, T9, (1985), 193-198.

The fractal dimension of the spectrum has been studied non rigorously by:

[129] Levitov, L. S.: Renormalization group for a quasiperiodic Schrödinger operator, *to appear in Europhys. Lett.*, (1988).

In a recent unpublished work, Bellissard, J., Iochum, B., Scoppola, E. & Testard, D., have studied rigorously this model and proved that:

(i) The spectrum of $H(x,\alpha)$ is independent of x for any α's.
(ii) If α is irrational and $\lambda \neq 0$ the spectrum of $H(x,\alpha)$ is a Cantor set.
(iii) The gap boundaries are continuous functions of α as long as α is irrational but they are discontinuous at each rational value of α.
(iv) The spectral measure is purely singular continuous, no states are localized.
(v) The spectrum and the wave functions can be computed from the case $\alpha = 0$ through a renormalization map similar to the map of §II-1 (eq.7) leaving also the same function $I(x,y,z)$ invariant.

II. The nearest neighbours model on a Penrose lattice with or without a magnetic field has been numerically studied in:

[130] Tsunetsugu, H., Fujiwara, T., Ueda, K., Tokohiro, T.: Eigenstates in a 2-dimensional Penrose tiling, *J. of Phys. Soc. Japan*, 55, (1986), 1420-23.

[131] Hatakeyama, T., Kamimura, H.: Electronic properties of a Penrose tiling lattice in a magnetic field, *Solid State Comm.*, 62, (1987), 79-83.

All these works exhibit evidence for Cantor spectrum.

III. Two recent works on the spectrum of the almost Mathieu equation (§II-1) have improved the result of theorem 5:

[132] Van Mouche, P.: The coexistence problem for the discrete Mathieu operator, *to appear in Comm. Math. Phys.*, who proves that the dense G_δ set in theorem 5 is actually independent of the coupling constant as long as it is not zero. And:

[133] Choi, M. D., Elliott, G., Yui, K.: Gauss polynomials and the rotation algebra, *Preprint Swansea* (1988) who give a wonderful proof that in the Harper equation (and also for the Almost Mathieu one if the coupling does not vanish) all the gaps which ought to be open are indeed open when α is rational; as a corollary they get an explicit dense set of irrational numbers for which the spectrum is a Cantor set.

IV. A one dimensional discrete Schrödinger operator with a quasiperiodic potential having two rationally independant frequencies has been studied rigorously by

[134] Sinai, Ya. G.: Anderson localization for the one dimensional difference Schrödinger operator with quasiperiodic potentials, *Proc. Int. Congress Math. Phys. Marseille 1986,* World Scientific, Singapore (1987), pp. 870-903.

[135] Chulaevsky, V. A., Sinai, Ya. G.: Anderson localization for a 1D discrete Schrödinger operator with two-frequency potentials, *subm. to Comm. Math. Phys.* (1988).
It is proved that provided the potential is given by a Morse C^2 function on $\mathbf{T}^3$ and the coupling constant is small enough, there is a set of positive Lebesgue measure in $[0,1]^{\times 2}$ such that for frequencies in that set, the corresponding Schrödinger operator has pure point spectrum with exponentially localized states. On the other hand the spectrum is a connected interval.

V. In a recent unpublished work, S. Kotani proved that if H is the hamiltonian on $l^2(\mathbf{Z}^2)$ given by $H\,\psi(n) = \psi(n+1) + \psi(n-1) + v(n)\,\psi(n)$ where $\mathbf{v} = (v(n))_{n\epsilon\mathbf{Z}}$ is a sequence with values in a *finite* set, then the spectral measure of H has an absolutely continuous component if and only if the sequence $\mathbf{v}$ is periodic.

VI. The model studied in §II-2 (theorem 15) has recently been investigated non rigorously by

[136] Keirstead, W. P., Ceccatto, H. A., Huberman, B. A.: Vibrational properties of hierarchical systems, *to appear in J. Stat. Phys.* (1988).

VII. The result of J. Avron & B. Simon [Avron (82)] quoted in §III-3 has also been obtained in

[137] Nenciu, G.: Stability of Energy Gaps Under Variations of the Magnetic Field, *Lett. Math. Phys.*, 11,(1986), 127-132.

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Dissipative Weakly Almost Periodic Functions

By John F. Berglund

Using the Bochner-von Neumann [2] definition of the almost periodic functions on a group, Eberlein [4] analogously defined the weak almost periodic functions. He then proceeded to show that the set of weak almost periodic functions enjoys many of the properties of the set of almost periodic functions: *e.g.*, that it is a uniformly closed linear space, indeed a C*-algebra, and, when the group is locally compact and abelian, it has an invariant mean and consists of uniformly continuous functions. Furthermore, all the functions "of interest" in harmonic analysis are weak almost periodic for a locally compact abelian group, at least; *viz.*, the almost periodic functions, the functions vanishing at infinity, and the Fourier-Stieltjes transforms.

In his second paper on the subject, Eberlein [5] considered the formal Fourier series

$$\sum_{\lambda\epsilon\Gamma} a_\lambda < t,\lambda >$$

associated with a weak almost periodic function f defined on a locally compact abelian group G with dual group Γ, where, if M denotes the unique invariant mean on the set $WAP(G)$ of weak almost periodic functions on G, then

$$a_\lambda = M[f(s) <-s, \lambda>].$$

Unlike the case for almost periodic functions, the Fourier series is not uniquely associated with the weak almost periodic function f. In fact, Eberlein showed that there is a unique decomposition $f=f_1+f_2$, where f_1 is almost periodic and has the same Fourier series as f and f_2 is weak almost periodic with $M(|f_2|^2) = 0$. We are concerned with this latter type of function, which we call *dissipative*.

A more general view of weak almost periodic functions was introduced by de Leeuw and Glicksberg in [3]. The Bochner-von Neumann definition of almost periodic is as follows: Let f be a continuous bounded complex-valued function on the topological group G. Define f_s by $f_s(t) = f(ts)$, $t\epsilon G$, and define $O(f) = \{f_s | s\epsilon G\}$. Then f is *almost periodic* if $O(f)$ is relatively compact in the norm topology of $C(G)$. Eberlein required that $O(f)$ be relatively compact in the weak topology to get the *weak almost periodic* functions. Clearly, not all the properties of a topological group are required for these definitions to make sense; in particular, neither group inverses nor joint continuity of the multiplication is required. Therefore de Leeuw and Glicksberg defined weak almost periodic functions on semigroups with separately continuous multiplication. They then proceeded to define and exploit the weak almost periodic compactification (w_S, S^W) of a semitopological semigroup S; that is S^W is a compact semitopological semigroup, $w_S: S \to S^W$ is a continuous homomorphism, $w_S(S)$ is dense in S^W and if $\psi : S \to T$ is a continuous homomorphism into a compact semitopological semigroup, then there is a continuous homomorphism $\psi^W : S^W \to T$ such that

$$\begin{tikzcd} S \arrow[r, "\psi"] \arrow[d, "w_S"'] & T \\ S^W \arrow[ur, "\psi^W"'] & \end{tikzcd}$$

commutes. A function f is weak almost periodic on S if and only if there is a continuous function $f^W \in C(S^W)$ such that $f = f^W \circ w_S$. Properties of the algebra of weak almost periodic functions are reflected as properties of the compactification. For example, $WAP(S)$ has an invariant mean if and only if S^W has a group as its minimal ideal $K(S^W)$. The dissipative functions f are, in that case, the ones for which $f^W|_{K(S^W)} \equiv 0$.

Soon after Eberlein's original paper [4], a question arose as to whether there were any functions $f \in WAP(G)$, where G is a locally compact abelian group, which were not uniform limits of Fourier-Stieltjes transforms of measures on Γ. Given the decomposition theorem, this amounts to the question of whether there are any dissipative functions on G which are not uniform limits of Fourier-Stieltjes transforms. In 1959, W. Rudin [7] gave an example of a dissipative function which cannot be approximated by Fourier-Stieltjes transforms. His example on the additive group $\mathbf{Z}$ of integers is the following:

$$f(m) = \begin{cases} e^{in \log n} & \text{if } m = k!n \ (k = 1, 2, 3, \ldots, 1 \le n \le k) \\ 0 & \text{otherwise.} \end{cases}$$

I am not myself interested in the Fourier-Stieltjes aspect of this problem, but in the behavior of dissipative functions.

Writing out the support of Rudin's function we have

$$\begin{aligned} \text{supp}\ (f) &= \{k!n \mid k = 1, 2, 3, \ldots; 1 \le n \le k\} \\ &= \{1, 2, 4, 6, 12, 18, 24, 48, 72, 96, 120, 240, \ldots.\} \end{aligned}$$

Note that there are larger and larger gaps in this set of integers. How typical is this of dissipative weak almost periodic functions? At first glance, one must conclude that it is not very typical since every function vanishing at infinity is weak almost periodic and adding one such to f will give us a weak almost periodic function with perhaps no gaps in its support. The following theorem shows that, taking into account the functions vanishing at infinity, the above pattern is typical.

Theorem. Let $(G,+)$ be a locally compact (not necessarily abelian) topological group, and let S be a closed, noncompact subsemigroup of G. Suppose that $WAP(S)$ has an invariant mean and that addition is continuous at every point of $S \times S^W$. Then the following statements about a function $f \in WAP(S)$ are equivalent:

(a) $f^W \equiv 0$ *on the minimal ideal* $K(S^W)$ *of* S^W.
(b) $M(|f|) = 0$, *where* M *is the unique invariant mean on* $WAP(S)$.
(c) *The zero function is in the weak closure of the orbit* $O(f)$.
(d) *For every* $\epsilon > 0$ *and every compact subset* K *of* S, *there is an element* $s \in S$ *such that*

$$\epsilon > \|R_s f\|_K = \sup\{|f(k+s)| : k \in K\}.$$

This is Theorem 3.4 of [1], and the proof is given there.

Although dissipative functions f are such that $|f|$ has large gaps in its "above ϵ" support, these gaps cannot be arbitrarily far apart, as the following theorem from [1] shows:

Theorem. Let $(G,+)$ be a locally compact topological group, and let S be a closed, noncompact, subsemigroup of G containing the identity 0. Suppose that $WAP(S)$ has an invariant mean and that addition is continuous at every point of $S \times S^W$. Suppose f is a dissipative weak almost periodic function on S. Let U_0 be a compact neighbourhood of the identity 0 of G and let $\epsilon > 0$. Then there is a compact neighbourhood $V = V(U_0, f, \epsilon)$ of 0 in G such that, for every $s \in S$, there exists $r \in S$ such that

$$(V+s) \cap (U_0 \cap S + r) \neq \emptyset$$

and

$$\epsilon > \|R_r f\|_{U_0} = \sup\{|f(t+r)| : t \in U_0 \cap S\}.$$

(Loosely speaking, this says that no matter where V is placed in S, it will intersect a set as big as U_0 where $|f|$ dips below ϵ;that is, these spots are relatively dense in S.)

The above theorems give us some reasonable information on the behavior of dissipative weak almost periodic functions, but a more desirable outcome would be an easily verified condition which would identify dissipative weak almost priodic functions among all bounded continuous complex-valued functions. Sufficient conditions were given by W. Rudin [7] and D. E. Ramirez [6], but they are far from necessary as has been shown by W. Ruppert [8]. Ruppert showed that functions f such as those produced by Rudin and Ramirez must vanish on $[S^W \backslash w_S(S)]^2$. On the other hand, dissipative functions need only vanish on $K(S^W)$ and, in general, $[S^W \backslash w_S(S)]^2 \neq K(S^W)$.

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Virginia Commonwealth University
Department of Mathematical Sciences
Box 2014, Richmond, VA 23284-0001
U.S.A.

On Maps which Preserve Almost Periodicity

By Pierre Eymard

If G is a topological group, let us denote by $AP(G)$ the set of all almost periodic functions on G, i.e.

$AP(G) = \{f: G \to \mathbf{C}$; the set of translates ${}_y f$, where $y \in G$, is relatively compact in $(\mathcal{CB}(G), \|\cdot\|_\infty)\}$.

Definition. Let G_2 and G_1 be two topological groups. A map $\rho: G_2 \to G_1$ is said to be *almost periodicity preserving* (a.p.p.) if, for every $f \in AP(G_1)$, one has $f \circ \rho \in AP(G_2)$.

In this lecture we shall see what exactly are the a.p.p. maps, under some particular hypothesis on the groups, and we shall give sketches of proofs in the two most classical cases: I) $G_2 = G_1 = \mathbf{R}$; II) $G_2 = G_1 = \mathbf{Z}$. These two examples are typical for the more general situation of connected groups, and discrete groups respectively. It turns out that the results are quite different in these two cases.

I. *Case* $G_2 = G_1 = \mathbf{R}$

Examples of maps $\rho: \mathbf{R} \to \mathbf{R}$ which are a.p.p. are:

1°) the group homomorphisms $\rho: x \to ax$, where a is a real constant;

2°) $\rho = h$, where h is a real-valued almost periodic function on $\mathbf{R}$.

Conversely one has the following:

Theorem 1: If $\rho: \mathbf{R} \to \mathbf{R}$ is a.p.p., then there exists $a \in \mathbf{R}$ and $h \in AP(\mathbf{R})$, such that $\rho(x) = ax + h(x)$.

Generalization. Suppose G_2 is an abelian locally compact *connected* group, and $G_1 = \mathbf{R}$; the same statement remains true, just replacing ax by $\sigma(x)$, where σ is any continuous group homomorphism of G_2 into $\mathbf{R}$. (Cf. [3]).

II. *Case* $G_2 = G_1 = \mathbf{Z}$

Let there be given an integer $p \geqq 1$, and for every $i = 0, 1, \ldots, p-1$, two integers a_i and b_i. For every $x \in \mathbf{Z}$, let us divide x by p, obtaining $x = pq + i$, where q is the quotient and i the rest, and put

$$\rho(x) = a_i q + b_i.$$

Definition. Such a $\rho: \mathbf{Z} \to \mathbf{Z}$ is called *piecewise affine* (of modulus p).

THEOREM 2. ρ: $\mathbf{Z} \to \mathbf{Z}$ is a.p.p. if and only if ρ is piecewise affine. (This result is implicitly in [2]).

III. *Generalizations to discrete groups* (Cf. [1])

Let G_2 and G_1 be two discrete groups, where G_1 is *abelian,* but not necessarily G_2. Let us denote by $\mathcal{F}(G_2)$ the set of all subgroups of G_2 which are invariant and *of finite index.*

Definition. ρ: $G_2 \to G_1$ is said to be *piecewise affine* if there exists a subgroup $H \in \mathcal{F}(G_2)$, some representatives x_0, $x_1,\ldots,x_{p-1}$ of the classes of G_2 modulo H, and for every $i = 0,1,\ldots,p-1$:

1°) a group homomorphism $\sigma_i = H \to G_1$;
2°) a fixed $b_i \in G_1$,

such that:

$$x = y\, x_i \quad , \quad y \in H \Rightarrow \rho\,(x) = \sigma_i(y) + b_i.$$

THEOREM 3. If G_2 is *finitely generated,* then ρ: $G_2 \to G_1$ is a.p.p. if and only if ρ is piecewise affine.

THEOREM 4. If G_2 is countable and *without proper invariant subgroups of finite index,* then ρ: $G_2 \to G_1$ is a.p.p. if and only if $\rho(x) = \sigma(x) + b$, where σ is a group homomorphism of G_2 into G_1, and $b \in G_1$.

Example of a G_2 such that $\mathcal{F}(G_2) = \{G_2\}$: the group of all permutations of $\mathbf{N}$ which act only on finitely many elements and are even on them; this group is countable and simple.

IV. *Sketch of proof of the Theorem 1* (Cf. [3])

Let ρ: $\mathbf{R} \to \mathbf{R}$ be a.p.p. The proof proceeds in 4 steps:

Step 1: ρ is uniformly continuous.

Step 2: there exists a constant C such that, for every $x \in \mathbf{R}$ and $y \in \mathbf{R}$, $|\rho(x+y) - \rho(x) - \rho(y)| \leqq C$.

Step 3: there exists a continuous group homomorphism σ: $\mathbf{R} \to \mathbf{R}$ (i.e. a dilation $x \to ax$) and a map ρ_1: $\mathbf{R} \to \mathbf{R}$ which is a.p.p. and *bounded* such that

$$\rho = \sigma + \rho_1.$$

Step 4: if ρ: $\mathbf{R} \to \mathbf{R}$ is a.p.p. and bounded, then $\rho \in AP(\mathbf{R})$.

The step 1 is quite natural. The step 4 is easy to prove, for instance passing through the Bohr compactification of **R.** We shall give proofs for steps 2 and 3.

Proof of Step 2: One can suppose $\rho(0) = 0$. Put $f(x) = e^{i\rho(x)}$ and $E = \{\tau \in \mathbf{R};\ \|f_\tau - f\|_\infty \leqq 1\}$. Since $f \in AP(\mathbf{R})$, there exists a compact interval K centered at 0, such that $\mathbf{R} = E + K$. For every $\tau \in E$, $x \in \mathbf{R}$,

$$\left|e^{i[\rho(x+\tau) - \rho(x)]} - 1\right| \leqq 1,$$

hence

$$\rho(x+\tau) - \rho(x) = m(\tau, x) + 2\pi\, k(\tau, x) \tag{1}$$

where $|m(\tau, x)| \leqq \frac{\pi}{3}$, and $k(\tau, x) \in \mathbf{Z}$. But for fixed τ, the first member of (1) is continuous in x on **R** (connected); hence $k(\tau, x) = k(\tau)$ does not depend on x. Notice that

$$C_1 = \sup\{|\rho(x) - \rho(y)|;\ x-y \in K\} \text{ is } < +\infty,$$

because ρ is uniformly continuous and K is compact.

Now for any $x \in \mathbf{R}$, $y \in \mathbf{R}$, choose $\tau \in E$ such that $\tau - x \in K$. We have

$$\begin{aligned} &|\rho(x+y) - \rho(x) - \rho(y)| \\ &\leqq |\rho(x+y) - \rho(\tau+y)| + |\rho(\tau+y) - \rho(y) - [\rho(\tau) - \rho(0)]| + |\rho(\tau) - \rho(x)| \\ &\leqq C_1 + |m(\tau, y) + 2\pi\, k(\tau) - m(\tau, 0) - 2\pi\, k(\tau)| + C_1 \\ &\leqq 2C_1 + \frac{2\pi}{3} = C. \end{aligned}$$

Proof of Step 3. Let M be a (Banach) invariant mean on $\mathcal{CB}(\mathbf{R})$. According to Step 2, for every fixed $y \in \mathbf{R}$, the function $x \to \rho(x+y) - \rho(x)$ is bounded continuous on **R**. We put:

$$\sigma(y) = M_x(\rho(x+y) - \rho(x)).$$

From the invariance property of the mean M results that $\sigma(y+z) = \sigma(y) + \sigma(z)$. Hence σ: $\mathbf{R} \to \mathbf{R}$ is a homomorphism, continuous because ρ is uniformly continuous. Evidently $\rho_1 = \rho - \sigma$ is a.p.p., and ρ_1 is bounded, since:

$$\begin{aligned} |\rho_1(y)| &= |\rho(y) - \sigma(y)| = |\rho(y) - M_x(\rho(x+y) - \rho(x))| = \\ &= M_x(\rho(y) + \rho(x) - \rho(x+y)) \leqq C\, M_x(1) = C. \end{aligned}$$

V. *Sketch of proof of Theorem 2*

Let $\mathbf{T}$ be the set of $z \in \mathbf{C}$, $|z| = 1$. For every $z \in \mathbf{T}$, the function $x \to z^{\rho(x)}$ is almost periodic on $\mathbf{Z}$; in particular: $\forall z \in \mathbf{T}$, $\exists$ an integer $k_z > 0$ such that $\sup_{x \in \mathbf{Z}} |z^{\rho(x)} - z^{\rho(x+k_z)}| \leqq 1$.

Applying the Baire theorem we see that:

$$\left.\begin{array}{l}\exists \text{ an integer } k_z > 0,\ \exists \text{ a set } S \subset \mathbf{T}, \text{ where } S \text{ is open and not empty,} \\ \text{such that } \forall z \in S,\ \forall x \in \mathbf{Z},\ |1 - x^{\rho(x+k)-\rho(x)}| \leqq 1.\end{array}\right\} \quad (2)$$

Necessarily the sequence $x \to \rho(x+k) - \rho(x)$ has only finite many values, because, if not, after H. Weyl, for a dense set of values of z, the sequence $x \to z^{\rho(x+k)-\rho(x)}$ should be dense in $\mathbf{T}$; but this is not true for $z \in S$ because of (2).

Choose $z \neq \sqrt[n]{1}$. The almost periodic function

$$x \longrightarrow z^{\rho(x+k)-\rho(x)}$$

takes only finitely many values; hence there exists $q \in \mathbf{Z}$, $q > 0$, such that $x \to \rho(x+k) - \rho(x)$ is constant on every class of $\mathbf{Z}$ modulo q. Working a little more, we can conclude that ρ is piecewise affine.

VI. *The crucial lemma for the proofs of Theorems 3 and 4.* (Cf. [1])

In the proof for $G_2 = G_1 = \mathbf{Z}$ we had the great simplification that every subgroup $\neq \{e\}$ of $G_2 = \mathbf{Z}$ is automatically of finite index. Under the more general conditions of Theorems 3 and 4, we need to prove directly that some subgroup of G_2, which occurs in the proof, is *in fact* of finite index. For that purpose I proved the following lemma, which perhaps has its own interest:

Finiteness lemma: Let $s = \{n_1 < n_2 < \ldots < n_k < \ldots\}$ be an increasing sequence of positive integers. Suppose that

$$\limsup_{n \to \infty} \frac{\upsilon(n)}{n} > 0,$$

where $\nu(n)$ is the number of integers $\leqq n$ in the sequence s.

If $\varepsilon > 0$, let $Z(\varepsilon) = \{z \in \mathbf{T};$ for every $n_k \in s$, $|1 - z^{n_k}| \leqq \varepsilon\}$.

Then there exists $\varepsilon_0 > 0$, not depending on s, such that $Z(\varepsilon)$ is *finite* for $\varepsilon \leqq \varepsilon_0$.

VII. *Interpretation in terms of Bohr compactification*

If G is a topological group, there exists a compact group $\overline{G}$ and a continuous group homomorphism $\beta\colon G \to \overline{G}$, such that $\beta(G)$ is dense in $\overline{G}$, and such that: $f \in AP(G) \Leftrightarrow$ there exists $\bar{f} \in \mathscr{C}(\overline{G})$ with $\bar{f} = f \circ \beta$.

Let G_2 and G_1 be two locally compact groups, with G_1 abelian. Then a map $\rho\colon G_2 \to G_1$ is a.p.p. if and only if there exists a continuous map $\bar{\rho}\colon \overline{G}_2 \to \overline{G}_1$ such that the diagram

$$\begin{array}{ccc} \overline{G}_2 & \xrightarrow{\bar{\rho}} & \overline{G}_1 \\ \beta_2 \uparrow & & \uparrow \beta_1 \\ G_2 & \xrightarrow{\rho} & G_1 \end{array}$$

is commutative.

This interpretation gives curious consequences of the theorems above, concerning the *analysis situs* of groups in their Bohr compactification. For instance:

Corollary 1. Let $\overline{\mathbf{Z}} = [(\mathbf{T})_d]^\wedge$ be the dual group of the discrete torus. If a continuous map of $\overline{\mathbf{Z}}$ into $\overline{\mathbf{Z}}$ carries $\mathbf{Z}$ into $\mathbf{Z}$, then the restriction of this map to $\mathbf{Z}$ is piecewise affine.

Corollary 2. Let G_2 and G_1 be discrete groups, where G_1 is abelian, and G_2 countable with $\overline{G}_2$ connected. If a map $\rho\colon G_2 \to G_1$ such that $\rho(e) = 0$ can be extended to a continuous map from $\overline{G}_2$ into $\overline{G}_1$, then necessarily ρ is a group homomorphism.

VIII. *A problem*

The hypothesis G_1 abelian in Theorems 3 and 4 is not very aesthetic. To determine the a.p.p. maps of the free group with two generators into itself seems to me an interesting problem to attack now.

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U.A. 750 associée au CNRS
Université de Nancy I
B.P. 239, F-54506 Vandœuvre les Nancy
France

Separated Solutions of Almost Periodic Differential Equations

By A. M. Fink

It is important to look for almost periodic solutions of differential equations since they tend to be the stable ones. In this paper I want to trace two threads of ideas which lead to almost periodic solutions of differential equations because of stability considerations. These two threads converge to give an elegant proof of Bohr's original theorem.

I recall Bohr's original Theorem [1] explicitly. If $F'(x) = f(x)$ and f is almost periodic, then F is almost periodic if and only if F is bounded. This is a theorem about solutions of the differential equation $y' = f(x)$. Although the original theorem was for f complex valued, it holds equally well for f a complex vector function.

The Bohr-Neugebauer Theorem [2] is about the solutions of the equation $x' = Ax + f(t)$ where f is a vector valued almost periodic function and A is a constant matrix. Again a solution x is almost periodic if and only if it is bounded. I will sketch the proof. By a change of variable we may assume that the matrix A is in Jordan canonical form. Then we look at a particular block,

$$x' = \begin{pmatrix} \lambda & 1 & \dots & 0 \\ 0 & \lambda & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & \lambda \end{pmatrix} x + \hat{f}.$$

The last equation is of the form $x' = \lambda x + f(t)$ with x a scalar. Proceeding up, the rest are of the form $x' = \lambda x + g(t)$ where x is a scalar, and g is a scalar almost periodic function.

If $\mathrm{Re}(\lambda) \neq 0$, then either

$$\int_{-\infty}^{t} e^{\lambda(s-t)} f(s)\,ds \quad \text{or} \quad \int_{t}^{\infty} e^{\lambda(s-t)} f(s)\,ds$$

is a bounded solution which is verified to be almost periodic by an obvious estimate. Since it is the only bounded solution the theorem holds for this component. If $\mathrm{Re}(\lambda) = 0$, then all solutions are bounded and almost periodic if and only if the solution

$$\int_{0}^{t} e^{\lambda(s-t)} f(s)\,ds$$

is bounded and hence almost periodic by Bohr's Theorem. Later we will give a different proof of this result to show how it fits in with other ideas. I remark here that this

Theorem also covers the n^{th} order scalar equation. For this observation one needs to know that for such equations, the solution x being bounded is equivalent to the vector $(x, x', x'', \ldots, x^{(n-1)})$ being bounded.

The Jordan form splits the system into systems living in invariant subspaces. Those coming from $\mathrm{Re}(\lambda) \neq 0$ exhibit the stability features. In those subspaces only one solution is bounded and almost periodic. The rest of the solutions are asymptotic (exponentially) to the almost periodic one either forward in time or backward in time. Equally important, they diverge exponentially either forward in time or backward in time. This situation is called an exponential dichotomy.

Favard attempted to generalize the above to the case when A is an almost periodic matrix. In order to describe his results elegantly we need to introduce some notation. For a real sequence $s = \{s_1, s_2, \ldots\}$ we define $T_s f(t) = \lim_i f(t + s_i)$ whenever this limit exists pointwise. If the limit is to exist in another sense, we will specify each time. We use T_s to denote translation along the sequence s.

The *hull* of an almost periodic function is the collection of functions g such that there is a sequence s for which $g = T_s f$ uniformly. The hull is denoted by $H(f)$ and is compact in $C(-\infty, \infty)$ in the uniform norm. For any $g \in H(f)$ we have $H(g) = H(f)$. Finally for sequences s and s', that s' is a subsequence of s is written as $s' \subset s$.

Along with the equation

$$x' = A(t)x + f(t) \tag{1}$$

we also consider all equations in the *non-homogeneous hull,* namely all equations of the form

$$x' = B(t)x + g(t) \tag{2}$$

where $B = T_s A$ and $g = T_s f$ uniformly, and all equations in the *homogeneous hull*

$$x' = B(t)x. \tag{3}$$

Favard's Theorem [3] is that if for every equation (3) all bounded non-trivial solutions satisfy $\inf_t |x(t)| > 0$, and there is a bounded solution of (1), then each equation (2) has an almost periodic solution.

This theorem includes the Bohr-Neugebauer result since those solutions of $x' = Ax$ which are bounded are almost periodic and therefore do not have zero infimum norm. A quick proof of this for almost periodic solutions of (3) can be given. If $x(s'_i) \to 0$, then take $s \subset s'$ so that $T_s B = C$, $T_{-s} C = B$, $T_s x = y$, $T_{-s} y = x$ all uniformly. It follows that y is a solution of $y' = Cy$ with $y(0) = 0$. Thus $y \equiv 0$ and a fortiori $x \equiv 0$. The simple equation $y'' + y = f(t)$ is an illustration. In phase space (y, y'), the solutions of the homogeneous equation have constant norm, e.g. $|(\cos t, -\sin t)| = 1$.

Favard's proof is to show that every equation in the non-homogeneous hull has a unique solution with minimum norm and that this implies its almost periodicity. We will reproduce this proof below with a modern twist.

Meanwhile we will trace a different thread of ideas which converge with that of Favard. Doss [4] observed that Bohr's Theorem on the primitive of an almost periodic function may be rephrased in the following way. If f is almost periodic then it is easy to see that function $F(t+h) - F(t) = \int_t^{t+h} f(s)ds$ is bounded, continuous and almost periodic. Now the left hand side can be considered without reference to integrals or primitives. He proved then that if F is a bounded continuous function such that for every h the difference $F(t+h) - F(t)$ is almost periodic, then F is almost periodic. It is easy to see that if the hypothesis holds for a dense set of h's and F is uniformly continuous then it holds for all h. A natural question arises, how many differences are required to be almost periodic for this theorem to hold?

Bochner [5] studied a general first order system which includes the possibility of delays and pure difference equations. He showed that if *one* non-trivial difference is almost periodic, and F is bounded and uniformly continuous, then F is almost periodic. In proving this theorem, Bochner derived a new necessary and sufficient condition for a function to be almost periodic. This condition is one which has become very useful in differential equations. *The condition is:* f is almost periodic if it is continuous and if for every pair of sequences t' and s', there are common subsequences (the same choice function for both) t and s such that $T_s T_t f = T_{s+t} f$ pointwise. The meaning of the left hand side is that $T_t f = g$ and $T_s g$ both exist. The usefulness of this criterion is that the condition is *pointwise*. Of course if f is almost periodic these hold uniformly.

If x is a bounded solution to a differential equation $x' = f(t,x)$ where f is almost periodic in t uniformly for x in compact sets, then from every pair of sequences s' and t' one can extract subsequences so that $T_s x = y$, $T_t y$, and $T_{t+s} x$ all exist uniformly on compact subsets of R. By taking further subsequences if neccessary, y will be a solution of the equation $x' = T_s f(t,x)$ and $T_t T_s x$ and $T_{t+s} x$ will both be solutions of the same almost periodic equation $x' = T_{t+s} f(t,x)$. To see how these ideas can be useful we will sketch the proof of Favard's Theorem. Recall that this proof will also prove Bohr's original theorem about primitives being almost periodic if they are bounded.

Sketch of Proof: First, if the set of bounded solutions of equation (2) is non-empty, then this set is a convex set which has a unique element with minimum norm. This is an argument using the parallelogram identity. For two distinct minimizing solutions x and y

$|\frac{x+y}{2}(t)|^2 + |\frac{x-y}{2}(t)|^2 = \frac{1}{2}|x(t)|^2 + \frac{1}{2}|y(t)|^2$. Since $\frac{x-y}{2}$ is a bounded solution of the

equation (3), the second term is larger than some $\delta > 0$. Taking supremums yields a contradiction. Call the minimum norm solution $x(B,g)$ for (B,g) in the hull of (A,f). If $T_s(A,f) \to (B,g)$ then by taking subsequences if necessary, $T_s x(A,f) = y$ is a solution of

(2) and $\|y\| \leqslant \|x(A,f)\|$. Now repeat the argument with the sequence $-s$. We get $T_{-s}y$ is a solution of (1) and $\|T_{-s}y\| \leqslant \|y\| \leqslant \|x(A,f)\|$. By uniqueness $T_{-s}y = x(A,f)$. It follows that $T_s x(A,f) = x(B,g)$, that is, the least norm solutions are translates of each other. Thus $T_t T_s x(A,f)$ and $T_{t+s} x(A,f)$ are both translates of a least norm solution and solutions of the same equation. By uniqueness they are the same and $x(A,f)$ satisfies Bochner's condition; it is almost periodic.

The above argument can be made with any functional L defined on solutions of equations in the hull of any almost periodic equation provided that 1) each equation has a unique minimizer of the functional L and 2) $L(T_s x) \leqslant L(x)$ for any solution x. The book of Amerio and Prouse [6] consists of giving examples of weak solutions of partial differential equations which minimize energy functionals. The main difficulty is to prove the existence of a unique minimizer.

A different situation where the Bochner criterion gives an elegant proof of almost periodicity is the case of a unique bounded solution. Specifically, suppose we have a differential equation $x' = f(t,x)$ such that for every equation in the hull, there is only one bounded solution. If x is such a solution, then $T_\alpha T_\beta x$ and $T_{\alpha+\beta} x$ are both the bounded solution of $x' = T_{\alpha+\beta} f(t,x)$ so are equal and the Bochner criterion shows x is almost periodic. It would seem that such a situation is too much to hope for, except there are nice examples where this is true. Moreover, if one replaces the word “bounded”, by “with values in a compact set K”, then the same argument applies.

It is instructive to consider specifically Bohr's original theorem for a real valued f. Suppose $y' = f(x)$ has the bounded solution $F(x)$. Then the function $G(x) = F(x) - \dfrac{\sup F + \inf F}{2}$ is the solution of the differential equation which is closest to zero in $C(-\infty,\infty)$, and $a \equiv \sup G = -\inf G$. Since it is uniformly continuous, for any sequence s' there is an $s \subset s'$ such that $T_s G$ is a solution of $y' = T_s f$, $-a \leqslant \inf T_s G$, and $\sup T_s G \leqslant a$. If strict inequality held, then translation by $-s$ would give a solution of $y' = f(x)$ whose norm is less than a. Consequently, for $K = [-a,a]$ there is a unique solution of each equation $y' = T_s f$ with values in K and G is almost periodic by the above argument. I think this is a very elegant argument.

Solutions which are isolated in a technical sense are called separated; x and y are separated solutions if there is a number d such that $|x(t) - y(t)| \geqslant d > 0$ for all t. This is the situation in Favard's Theorem. Amerio [7] generalized this to the non-linear case. The hypotheses need apply to all equations in the hull. Suppose that in some compact set K there are only finitely many solutions and that they are separated, then they are all almost periodic.

A property that implies the separated property is uniform stability. Uniform stability is a strong continuity with respect to initial conditions. A solution x is uniformly stable on $[a, \infty)$ if for a given $\varepsilon > 0$ there is a $\delta > 0$ such that if y is a solution such that $|x(t_0) - y(t_0)| < \delta$, then $|x(t) - y(t)| < \varepsilon$ for all $t \geqslant t_0 \geqslant a$. Uniform stability of a solution implies that it is separated on intervals of the form $(-\infty, b]$. If $|x-y|\ (0) = \varepsilon$, then $|x-y|\ (t) \geqslant \delta(\varepsilon/2)$ for $t<0$ else $|x-y|\ (0) < \varepsilon/2$. If t'_n is a sequence such that $T_{t'_n} f = f$ and $t'_n \to -\infty$, take a subsequence $t \subset t'$ such that $T_t x$ and $T_t y$ are solutions of $x' = f(t,x)$. Then $|T_t x - T_t y| \geqslant \delta\ (\frac{\varepsilon}{2})$ on the real line.

The various concepts of stability and the relationships with almost periodicity have been studied intensly by Seifert, Yoshizawa, Fink and others including researchers in the USSR. The book by Levitan and Zhikov [8] outlines these developments in the USSR, while Fink [9] discusses all of the above ideas. A more complete discussion of all aspects of stability and almost periodicity is given in Yoshizawa [10].

Some specific equations to which the above ideas apply can also be found in [9]. One of the more remarkable results is that of Frederickson and Laser [11]. The equation $x'' + f(x)\ x' + x = p(t)$ with almost periodic p has an almost periodic solution if and only if $F(\infty) - F(-\infty) > \pi\beta$ where $F(x) = \int_0^x f$ and $\beta = \max_s M\{p(t) \sin (s-t)\}$. The solution is uniformly quasi-asymptotically stable in the large.

For scalar equations $x' = f(x,t)$, if there is a bounded uniformly stable solution on $[0,\infty)$ then there is an almost periodic solution. If f is monotone in x and there is a bounded solution on $[0,\infty)$ then there is an almost periodic solution. Each of these solutions is a unique solution contained in some compact set. The compact set is obtained by separation in the first case and by minimization of the oscillation function in the second.

A second order example is the equation $x'' = m(x)x' + g(x) + e(t)$ where e is almost periodic. Define $M(x) = \int_0^x m$ and

$$\phi(u,v) = \begin{cases} \dfrac{M(u+v)-M(u)}{v} & v \neq 0; \\ m(u) & v = 0; \end{cases} \qquad h(u,v) = \begin{cases} \dfrac{g'(u+v)-g'(u)}{v} & v \neq 0. \\ g'(u) & v = 0. \end{cases}$$

If there is an $a < b$ for which

$$g(a) + e(t) \leqslant 0 \leqslant g(b) + e(t) \text{ holds for all } t,\ g'(t) > 0,$$

and there is a λ so that

$$(\phi(u,v) - \lambda)^2 - 4h(u,v) \leqslant 0 \text{ for } u,\ u+v \in [a,b],$$

then there is an almost periodic solution. This is a unique solution with values in $[a,b]$ and is stable.

A slightly different set of sufficient conditions is illustrated by the equation

$$x'' + f(x)\ x' + g(x) = k\ p(t)$$

where p is almost periodic. Let $F(x) = \int_0^x f$, $g(0) = 0$, g' exist and satisfy $0 < g'(x) < \beta$ and $f(x) \geqslant \alpha$ where $\beta < \alpha^2$. Suppose one can find $c < d$ so that $g(c) = -k$ and $g(d) = k$ and $a < b$ such that $k < \min\{\,[F(d) - F(c)]f(x) + g(-b), [F(-a) - F(-b]f(x) - g(d)\,\}$ on $[-b, d]$, then there is a unique bounded solution which is uniformly stable and almost periodic.

Finally, I mention that the notions of stability and separatedness have their counterparts in the abstract theory of dynamical systems. The use of dynamical systems for non-autonomous equations was inaugurated by Miller [12].

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Iowa State University
Ames, IA 50011
U.S.A.

Value-distribution Theory for Holomorphic Almost Periodic Functions

By SIGURÐUR HELGASON

1. *Introduction*

I am deeply indebted to the organizers for inviting me to participate in this conference on the occasion of the 100th anniversary of Harald Bohr.

My student years here in Copenhagen happened to coincide with the last 5 years of Bohr's life. Although many years have passed I still have a vivid memory of his inspiring lectures and of his personal kindness.

Today I am going to talk about some work of my own from this time; this was a response to a prize question posed by the University for 1950 concerning holomorphic almost periodic functions. This work was inspired by papers by Bohr and by Jessen, together with the works of the Finnish mathematical school (primarily Rolf Nevanlinna and Lars Ahlfors) on value-distribution theory of meromorphic functions. The new results are described in §§ 4-5.

First I will describe some background material. Here I am indebted to Prof. B. Fuglede and Prof. H. Tornehave for some informative references. Bohr's early work on Dirichlet series and the Riemann zeta-function led him to the theory of almost periodic functions. While the principal results of his theory of almost periodic functions on **R** have to some extent been absorbed in the theory of continuous functions on compact abelian groups, his theory of holomorphic almost periodic functions [2] has retained its independence and its charm.

A holomorphic function $f(s)$ in a vertical strip (α, β): $\alpha < \operatorname{Re} s < \beta$ is said to be *almost periodic* if to each $\epsilon > 0$ there exists a number $l = l(\epsilon)$ such that each interval $t_0 < t < t_0 + l$ of length l contains a number τ such that

$$|f(s+i\tau) - f(s)| \leq \epsilon$$

for all s in the strip. (Here α and β are allowed to be infinite.) In other words, if $s = \sigma + it$, $\alpha < \sigma < \beta$, the function $t \to f(\sigma + it)$ is almost periodic on **R** and uniformly so for $\alpha < \sigma < \beta$.

To each such function f one can associate its Dirichlet series

$$f(s) \sim \sum_n A_n e^{\Lambda_n s} \qquad \Lambda_n \in \mathbf{R}, \tag{1}$$

which determines it uniquely. Here

$$A_n = \mathcal{M}_t\left(f(\sigma + it)e^{-\Lambda_n(\sigma + it)}\right) \tag{2}$$

where $\mathcal{M}$ is the mean value

$$\mathcal{M}(\varphi) = \lim_{S-R\to\infty} \frac{1}{S-R} \int_R^S \varphi(t)dt.$$

For φ almost periodic this limit does indeed exist and then the holomorphy of f implies that A_n is indeed independent of σ. A uniformly convergent Dirichlet series (say $\varsigma(s) = \Sigma_1^\infty e^{-(\log n)s}$ for Re $s > 1 + \epsilon$) is almost periodic; on the other hand, to an almost periodic function $f(s)$ in a strip can be associated a sequence $f_p(s)$ of exponential polynomials $\Sigma_n A_n^{(p)} \exp(\Lambda_n^{(p)} s)$ which converge to $f(s)$ uniformly in any closed substrip ($\alpha_1 \leq$ Re $s \leq \beta_1$, where $\alpha < \alpha_1 < \beta_1 < \beta$).

The original Dirichlet series

$$\sum_1^\infty \frac{a_n}{n^s} = \sum_1^\infty a_n e^{-(\log n)s}$$

were generalized to series of the form

$$\sum_n a_n e^{\Lambda_n s} \qquad \Lambda_1 > \Lambda_2 > \ldots \tag{3}$$

and both at the beginning and the end of his career Bohr investigated problems of convergence, summability etc. for such series (3). It is therefore worth stressing that in (1) the order of the exponents is unrestricted.

2. *Result of Jessen. The Jensen Function*

With the Riemann zeta-function as motivation it becomes a problem of interest to study the distribution of zeros of a function $f(s)$ almost periodic in a strip (α, β). For such functions f the basic general results were obtained by Jessen [6]. He showed the existence of the limit

$$\varphi_f(\sigma) = \mathcal{M}_t(\log|f(\sigma + it)|) = \lim_{S-R\to\infty} \frac{1}{S-R} \int_R^S \log|f(\sigma + it)|\, dt \tag{1}$$

(in spite of the fact that f may have zeros) and proved that it is a *convex* function of σ. Jessen's principal result is the following theorem. If $\alpha < \alpha' < \beta' < \beta$ let $n(\alpha', \beta'; R,S)$ denote the number of zeros of f in the rectangle $\alpha' < \sigma < \beta'$, $R < t < S$ counted with multiplicity.

THEOREM 2.1. *If φ_f is differentiable at α' and β' then the density of zeros*

$$H(\alpha', \beta') = \lim_{S-R\to\infty} \frac{n(\alpha', \beta'; R, S)}{S-R}$$

exists and

$$H(\alpha', \beta') = \frac{1}{2\pi}\{\varphi_f'(\beta') - \varphi_f'(\alpha')\}.$$

If the function $t \to f(\sigma + it)$ has a fixed period p it turns out that this is equivalent to the classical Jensen formula in complex function theory; for this case the function φ_f is a piecewise linear function. Jessen called φ_f *the Jensen function* for f.

Indication of proof. First we assume that the boundary of the rectangle $\alpha' < \sigma < \beta'$, $R < t < S$ contains no zero of $f(s)$. Then by standard complex variable theory.

$$2\pi\, n(\alpha', \beta'; R, S) =$$

$$\int_R^S \frac{f'(\beta' + it)}{f(\beta' + it)}dt - \int_R^S \frac{f'(\alpha' + it)}{f(\alpha' + it)}dt - i\int_{\alpha'}^{\beta'} \frac{f'(\sigma + iR)}{f(\sigma + iR)}d\sigma + i\int_{\alpha'}^{\beta'} \frac{f'(\sigma + iS)}{f(\sigma + iS)}d\sigma.$$

Consider the vertical segments $\alpha' + it$ $(R \le t \le S)$ and $\beta' + it$ $(R \le t \le S)$. We can find a simply connected region Ω containing both of these segments and no zeros for f. We can then define the logarithm $\log f(s)$ in Ω, divide the relation above by $S-R$ and let it tend to ∞. We can restrict the R and S in such a way that the two last terms above give no contribution in the limit. The identity in Theorem 2.1 follows by taking real parts. The restriction on α' and β' is then removed by a continuity argument.

It is now an interesting problem to characterize the convex functions $\varphi(\sigma)$ which arise as Jensen functions φ_f for suitable almost periodic $f(s)$. This question was investigated by Buch [4] whose results imply for example that any convex function which is not linear on any interval arises in this fashion. A complete characterization of the φ_f was given by Jessen and Tornehave [7], § 112. It implies for example that a convex function $\varphi(\sigma)$, $\alpha < \sigma < \beta$, having infinitely many intervals of linearity in a compact subinterval of (α, β) cannot be a Jensen function φ_f if the slopes $\varphi'(\sigma)$ in these intervals are linearly independent over the rational numbers.

3. *Normal Almost Periodic Functions*

Already in his original paper [2] Bohr made a special investigation of almost periodic functions $f(s)$ in a half-plane $(-\infty, \beta)$ and expressed their behaviour near $\sigma = -\infty$ in terms of their Dirichlet expansion

$$f(s) \sim \sum_n A_n e^{\Lambda_n s} \qquad (A_n \neq 0). \tag{1}$$

In [3] he singled out the so-called *normal almost periodic functions* $f(s)$ for which among the nonzero exponents Λ_n the smallest one exists. These have the following property:

Given any $a \in \mathbf{C}$, *there exists a half-plane* $(-\infty, \sigma_a)$ *which contains no a-point for* $f(s)$ (*i.e. a zero of* $f(s) - a$).

Let us for a moment view such a function $f(s)$ via the substitution $s = \log z$ as a function $\varphi(z)$ on a piece $0 < \rho < \rho_0$ of the Riemann surface of $\log z$. The series (1) then becomes a generalized Laurent series

$$\varphi(z) \sim \Sigma A_n z^{\Lambda_n} \qquad A_n \neq 0. \tag{2}$$

Let us for simplicity assume the normalizing property that the lowest nonzero exponent, say Λ_0, is > 0. Bohr showed in [3] that the inverse function is also normal almost periodic. I have proved in [5] that a similar statement can be made about the composition of two normal almost periodic functions (having the above normalizing property).

4. *Value-distribution Theory. The first Fundamental Theorem*

Consider a fixed $\beta \leq \infty$ and let $z = f(s)$ be normal almost periodic in $\{-\infty, \beta\}$ (that is normal almost periodic in any substrip $(-K, \beta_1)$ where $\beta_1 < \beta$). We apply stereographic projection of the z-plane $\mathbf{C} \cup \{\infty\}$ onto the Riemann sphere S with diameter 1, tangential to the z-plane at $z = 0$. Given $z_1, z_2 \in \mathbf{C} \cup \{\infty\}$ the (chordal) distance of the corresponding points on S is given by

$$k(z_1,z_2) = \frac{|z_1 - z_2|}{\sqrt{(1+|z_1|^2)\,(1+|z_2|^2)}}$$

so the arc-length element $d\sigma$ on S is

$$d\sigma = \frac{|dz|}{1+|z|^2}.$$

For any $a \in \mathbf{C} \cup \{\infty\}$ let Λ_a denote the smallest exponent in the Dirichlet expansion of $f(s) - a$. (Here $f(s) - \infty$ is to be understood as $1/f(s)$ which is also normal almost periodic). Let A_a denote the corresponding coefficient. We now introduce a quantity which measures how well the function $f(s)$ approximates the value a on the line $\mathrm{Re}\, s = \sigma$. Put

$$M(\sigma,a) = \mathscr{M}_t\{-\log(k(f(\sigma+it), a))\} + [\log k(f(-\infty),a)], \tag{1}$$

where the remainder term is

$$[\log k(f(-\infty),a)] = \begin{cases} \log k(f(-\infty),a), & f(-\infty) \neq a \\ \log \dfrac{|A_a|}{1+|a|^2} & f(-\infty) = a \neq \infty \\ \log\ |A_\infty| & f(-\infty) = a = \infty. \end{cases}$$

The existence of the integral in (1) is clear from the existence of (1) § 2. Next we put $n(\sigma,a;R,S)$ = the number of a-points (with multiplicity) of $f(s)$ in the rectangle $-\infty < \tau < \sigma$, $R < t < S$ (with $s = \tau + it$).

$$N(\sigma,a) = \lim_{S-R\to\infty} \frac{2\pi}{S-R}\int_{-\infty}^{\sigma} n(\tau,a;R,S)\, d\tau + n(-\infty,a;-\infty,\infty)\sigma, \tag{2}$$

where

$$n(-\infty,a;-\infty,\infty) = \mathrm{Max}(\Lambda_a,0).$$

The existence of the last limit is easily established by means of tools used in the proof of Theorem 2.1. The function $N(\sigma,a)$ is taken as a measure for the number of a-points of f in the half-plane $\mathrm{Re}\, s < \sigma$. Note that the remainder term in (2) appears only if $a = \lim_{\sigma\to-\infty} f(s)$.

Theorem 4.1. *If $f(s)$ is normal almost periodic in $\{-\infty, \beta\}$ then the sum*

$$M(\sigma,a) + N(\sigma,a) = T(\sigma) \tag{3}$$

is independent of a. Also

$$T(\sigma) = \lim_{S-R\to\infty} \frac{2\pi}{S-R}\int_{-\infty}^{\sigma} A_{RS}(\tau)d\tau, \tag{4}$$

where $A_{RS}(\tau)$ *is the area of the Riemann surface* $V_{R,S}(\tau)$ *over the Riemann sphere onto which the function* f *maps the rectangle* $-\infty < \rho < \tau$, $R < t < S$.

The fact that the sum $M(\sigma, a) + N(\sigma, a)$ is independent of a is an analog of Nevanlinna's first fundamental theorem for meromorphic functions. It implies that if $N(\sigma,a)$ is small that is, if $f(s)$ has few a-points, then the approximation term $M(\sigma,a)$ is large and vice-versa. The function $T(\sigma)$ is called *the characteristic function*. The geometric interpretation (4) of $T(\sigma)$ is an analog of a similar interpretation for the classical (periodic) case given by Ahlfors [1] and Shimizu [9].

The proof of Theorem 4.1 proceeds along lines similar to the classical theory (Nevanlinna [8], VI, § 3) but requires in addition some tools utilized in the proof of Theorem 2.1. A brief indication follows. Let $A \in \mathbf{C}$ and put

$$w(s) = A + \frac{1}{f(s)}, \qquad v(s) = \log(1 + |w(s)|^2).$$

We use Gauss' formula

$$\int_\Gamma \frac{\partial v}{\partial n}\, dl = \iint_\Omega \Delta v \, d\tau dt \tag{5}$$

on a region Ω which is the rectangle $\sigma_0 < \tau < \sigma$, $R < t < S$ with small disks removed having the zeros of f as the centers. Here Γ is the boundary of Ω (with the appropriate orientation), dl the arc element on Γ, Δ the Laplacian in the (τ,t) variables and $\partial/\partial n$ the outgoing normal derivative.

The proof now proceeds along the following steps.

(i) We use the Laurent series of $1/f(s)$ around each zero of $f(s)$ to estimate the contributions to the left hand side of (5) of the circular parts of Γ. Then we let the radii of the disks considered tend to 0.

(ii) By direct computation

$$(\Delta v)(s) = 4\frac{|w'(s)|^2}{(1+|w(s)|^2)^2}.$$

Viewing $w(s)$ as a map from the s-plane to the Riemann sphere lying on the w-plane we have

$$\frac{d\sigma}{|ds|} = \frac{1}{1+|w(s)|^2}|w'(s)|.$$

Thus if $A_{RS}(\tau)$ is the area function (for the function $w(s)$) we have (with $s = \tau + it$)

$$A_{RS}(\sigma) = \int_R^S dt \int_{-\infty}^{\sigma} \frac{|w'(s)|^2}{(1+|w(s)|^2)^2} d\tau.$$

(iii) In (5) we divide by $S-R$ and let $S-R \to \infty$ through special values of S and R, such that on the corresponding horizontal lines $\tau + iR$, $\tau + iS$ $(\sigma_0 < \tau < \sigma)$ $f(s)$ is bounded away from 0. Then the horizontal pieces of the boundary Γ in (5) give no contribution to the limit. The normal derivatives in (5) can be pulled outside the integral as $\partial/\partial\sigma$.

(iv) Now let $\sigma_0 \to -\infty$ in (5) and then integrate with respect to σ from $-\infty$ to σ. Considering the behaviour of $w(s)$ as $\sigma \to -\infty$ we obtain after some manipulation the formula

$$\begin{aligned} &\mathcal{M}_t \left(\log \sqrt{1+|w(\sigma+it)|^2}\right) + N(\sigma,0) = \\ &= \lim_{S-R\to\infty} \frac{2}{S-R} \int_{-\infty}^{\sigma} A_{RS}(\tau) d\tau + \log \sqrt{1+|w(-\infty)|^2}. \end{aligned} \tag{6}$$

The last term should be replaced by $\log(1/|A_0|)$ in case $w(-\infty) = \infty$.

(v) Consider a fixed $a \in \mathbf{C}$ and the function

$$w_1(s) = \frac{1+\bar{a}f(s)}{f(s)-a} = \bar{a} + \frac{1}{(f(s)-a)\ (1+|a|^2)^{-1}}.$$

The values of $w_1(s)$ are obtained from the values of $f(s)$ by rotation of the sphere so A_{RS} is the same for w_1 and for f. Also

$$(1 + |w_1(s)|^2)^{-1} = k(w_1(s), \infty) = k(f(s),a)$$

so when (6) is used on w_1 we do obtain Theorem 4.1.

5. *The Second Fundamental Theorem. Applications*

While the first fundamental theorem expresses the constancy of the *total affinity* $M(\sigma,a) + N(\sigma,a)$ of $f(s)$ to the value a the second fundamental theorem will show that for most a $N(\sigma,a)$ is the principal component. This is based on an estimate of the sum $\sum_{v=1}^{p} M(\sigma,a_v)$ for arbitrary distinct $a_1, \ldots a_p$.

Motivated by the classical (periodic) theory we consider the number $n_1(\sigma, a; R, S)$ of multiple roots in the equation $f(s) - a = 0$ in $-\infty < \tau < \sigma$, $R < t < S$, such that a k-fold root is only counted $(k-1)$ times. We also put $n_1(-\infty, a; -\infty, \infty) = \mathrm{Max}(\Lambda_a', 0)$, where Λ_a' is the smallest exponent in the expansion of $f'(s) - a$. Clearly $\sum_a n_1(\sigma, a; R, S)$ is the number of zeros for $f'(s)$ in the rectangle indicated. Then we put

$$N_1(\sigma,a) = \lim_{S-R\to\infty} \frac{2\pi}{S-R}\int_{-\infty}^{\sigma} n_1(\tau,a;\, R,S)d\tau + n_1(-\infty,\, a;-\infty,\infty)\sigma \ , \tag{7}$$

and by the remark above, $\sum_a N_1(\sigma,a)$ is bounded by the function $N(\sigma, 0)$ taken for the derivative f'. In the next theorem (the analog of Nevanlinna's second fundamental theorem) we distinguish between the two cases: β finite and $\beta = \infty$.

THEOREM 5.1. I. *Let $f(s)$ be normal almost periodic in $\{-\infty,\infty\}$ and $a_1, \ldots, a_p$ arbitrary distinct complex numbers. The inequality*

$$\sum_{\nu=1}^{p} M(\sigma, a_\nu) + \sum_a N_1(\sigma,a) \le T(\sigma) + O(\log T(\sigma)) + O(\log|\sigma|) \tag{8}$$

holds for all σ except on a set of σ of finite measure.
II. *Let $f(s)$ be normal almost periodic in $\{-\infty, 0\}$ and $a_1, \ldots, a_p$ any distinct complex numbers. Then inequality* (8) *holds with* $\log|\sigma|$ *replaced by* $\log(1/(1-e^{\sigma}))$ *and the inequality holds for all $\sigma < 0$ except for at set of σ over which the integral of $e^{\sigma}(1-e^{\sigma})^{-1}$ is finite.*

In the proof of this theorem the passage from periodic functions to almost periodic functions gives rise to certain technical difficulties. The proof is therefore too complicated to describe here in detail. Instead, I will show how the theorem implies the analog of Nevanlinna's defect relation.

Application. Let f be normal almost periodic in $\{-\infty, \beta\}$. For each $a \in \mathbf{C}$ we define the *defect* by

$$\delta(a) = 1 - \overline{\lim_{\sigma\to\beta}} \frac{N(\sigma,a)}{T(\sigma)}$$

and the *ramification index*

$$\nu(a) = \lim_{\sigma\to\beta} \frac{N_1(\sigma,a)}{T(\sigma)} .$$

In the case when f is a nonconstant normal almost periodic function in $\{-\infty, \infty\}$ it is easily deduced from Theorem 4.1 that $\underline{\lim}_{\sigma\to\infty} T(\sigma)/\sigma > 0$. From Theorem 5.1 we can therefore deduce the following result.

THEOREM 5.2. Let $f(s)$ *be nonconstant and normal almost periodic in* $\{-\infty, \infty\}$. *Then the defect* $\delta(a)$ *and the ramification index* $\nu(a)$ *are strictly positive for at most countably many a and*

$$\sum_a \delta(a) + \sum_a \nu(a) \leq 1.$$

The defect $\delta(a)$ is a measure for how rarely f takes the value a. If a is omitted by $f(s)$ altogether in $-\infty \leq \operatorname{Re} s < \infty$ then $\delta(a) = 1$ so we deduce from $\sum_a \delta(a) \leq 1$ that there can be at most one such value a.

For the case $\beta = 0$ we obtain similarly the following result.

THEOREM 5.3. *Let* $f(s)$ *be normal almost periodic in* $\{-\infty, 0\}$ *and assume*

$$\varliminf_{\sigma \to 0} \frac{\log\left(\frac{1}{1-e^{\sigma}}\right)}{T(\sigma)} = 0.$$

Then

$$\sum_a \delta(a) \leq 1.$$

Again this implies that f omits at most one value in the strip $-\infty \leq \operatorname{Re} s < 0$.

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Department of Mathematics
MIT 2-182
Cambridge, MA 02139
U.S.A.

Around Bohr's thesis

By JEAN-PIERRE KAHANE

Bohr's thesis [2] deals with ordinary Dirichlet series

$$\sum_1^\infty a_n n^{-s}. \tag{1}$$

I shall try to describe the mathematical context at the time when he wrote the thesis, 1909, then – very shortly – his main results (from 1909 to 1951) and the problems he left open, then the role of series

$$\sum_1^\infty \mp n^{-s} \tag{2}$$

and the present state of Bohr's problems. Short proofs of classical things are given at the end.

1. *Before 1909*

One century ago, three days after Harald Bohr was born, J. L. W. V. Jensen, a telephone engineer from Copenhagen, presented a paper at the French Academy of Sciences, entitled "Sur la fonction $\zeta(s)$ de Riemann" [14]. H. Bohr quotes it in his first paper, in his thesis, and a number of times: obviously it has been a source of inspiration for him. When he evokes Jensen, he says he was "one of the most gifted mathematicians our country has ever produced".

Jensen was interested i Dirichlet series. He introduced the basis formula

$$|e^{-\lambda_n s} - e^{-\lambda_{n+1} s}| \le \frac{|s|}{|\sigma|} (e^{-\lambda_n \sigma} - e^{-\lambda_{n+1}\sigma})$$

which allowed him to prove that, if a Dirichlet series $\Sigma\, a_n e^{-\lambda_n s}$ converges at a point (say, 0), it converges uniformly on every compact set which lies strictly at the right [13]. His paper on $\zeta(s)$ was motivated by two reasons: first, give a simple proof, not using the functional equation of Riemann, that $(1-s)\zeta(s)$ is an entire function; then, taking for granted what Stieltjes had claimed two years before [27] – namely that he had a proof of the Riemann hypothesis – investigate the location of the first zeros on the critical line.

Stieltjes was also interested in Dirichlet series. In order to derive from the Riemann hypothesis – which he thought he had proved – results on prime numbers he needed multiplication of Dirichlet series. And he stated a curious result: that the product of two Dirichlet series

$$\Sigma a_n n^{-s} \quad , \quad \Sigma b_n n^{-s},$$

(namely $\Sigma c_n n^{-s}$ with $c_n = \sum_{mp=n} a_n b_p$) converges for $\sigma > \frac{1}{2}$ if the two first converge for $\sigma > 1$ [28].

In the last decade of the 19th century, Dirichlet series was an interesting topic. The main work was Cahen's thesis, in France, with a number of formulas for the coefficients, the abscissa of convergence, etc. [8]. Also, with a wrong statement, namely that the theorem of Stieltjes on multiplication of Dirichlet series could be improved, replacing $\sigma > \frac{1}{2}$ by $\sigma > 0$ in the conclusion. Hadamard [9] and de la Vallée Poussin [29] proved that $\zeta(s)$ has no zero on $\sigma = 1$ and derived the prime number theorem.

However, around 1900, there was a decline of interest for Dirichlet series, together with a renewal of interest for *Fourier series*, mainly because of the Lebesgue integral and the Fejér summation theorem.

Then, suddenly, at the time Harald Bohr began to work, a number of first class mathematicians turned again to Dirichlet series. In 1907 and 1908, there were several papers of Landau [17], [18], a short article by Hadamard [10], an extensive study by O. Perron [20], and the important thesis of Schnee [25]. Landau published the first proof of Stieltjes's statement and observed that Cahen had been wrong on multiplication of Dirichlet series. Schnee, among other results, proved that a Dirichlet series converges for Re $s > \sigma_0$ whenever the function $f(s) = \sum_0^\infty a_n e^{-\lambda_n s}$ exists for Re s large and can be extended as a function of order 0 in Re $s > \sigma_0$, that is

$$f(\sigma + it) = O(|t|^\varepsilon) \quad (\sigma > \sigma_0, |t| \to \infty)$$

for each $\epsilon > 0$ [25] [26].

In 1908 again, Lindelöf proved his famous convexity theorem about the order of a function. With Bohr's notations, if we write

$$\mu(\sigma) = \inf \{a \mid f(\sigma + it) = O(|t|^a)\} \quad (t \to \infty)$$

when $f(s)$ is holomorphic in the strip $a < \sigma < b$ $(s = \sigma + it)$, then $\mu(\sigma)$ is a convex function [19].

In 1909, Marcel Riesz published three important notes in Comptes-Rendus, all of them on Dirichlet series [22], [23], [24].

2. *Bohr's results and problems (1909–1950)*

1909 is the year Harald Bohr writes his thesis. It begins with a note aux Comptes-Rendus, his first paper, *11th of January*, 1909, "Sur la série de Dirichlet" [1]. And the year ends with the approval of the thesis, signed by the dean on *December 31, 1909*. In between, he writes also a paper for the Göttinger Nachrichten, on the summability of Dirichlet series, the topic of his first note [3]. His starting point is like this: the series $\Sigma\,(-1)^n\,n^{-s}$, which represents $\zeta(s)\;(1-2^{1-s})$, is summable by the Cesàro process of order r when $\sigma \geq -r$, therefore represents an entire function (a still shorter proof than Jensen's).

Actually, after 1909, his main interest shifted to the ζ–function, then, after 1920, to the theory of almost periodic functions. Nevertheless, his last papers, around 1950, all deal with the problems he considered in his thesis [4], [5], [6], [7].

I shall review at the same time what he did in his thesis and the improvements he gave in the 1950's.

A: The convergence problem

Let σ_a be the abscissa of absolute convergence and σ_c the abscissa of convergence of an ordinary Dirichlet series. Then

$$\sigma_c \leqslant \sigma_a \leqslant \sigma_c + 1 \text{ and } \mu(\sigma_a) = 0 \text{ (obvious)}$$

$$\mu(\sigma_c) \leqslant 1 \text{ (Jensen)}$$

$$\mu(\sigma) = 0 \Rightarrow \sigma_c \leqslant \sigma \text{ (Schnee)}$$

Is it possible to improve, that is, to obtain more information on σ_c from the order function $\mu(.)$ or more information on $\mu(.)$ from the abscissa of convergence σ_c? The answer is negative and it is provided by two examples: a lacunary series of the form

$$\sum_1^\infty \left(p_n^{-s} - (p_n + 1)^{-s}\right) \tag{3}$$

gives figure 1, and a more complicated example figure 2.

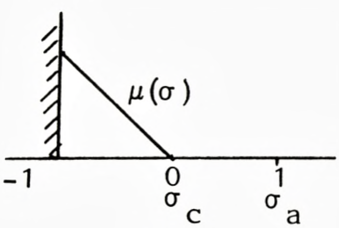

Fig. 1.

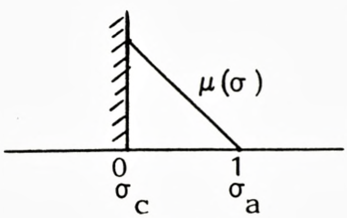

Fig. 2

Figure 2 answers the Stieltjes-Cahen-Landau multiplication problems, because

$$\mu(\sigma\,;f^2) = 2\mu(\sigma\,;f)$$

therefore, with f as in figure 2,

$$\sigma_c(f^2) \geq \frac{1}{2}$$

by Jensen. Bohr was very happy of this discovery and came back to the multiplication problem later [5] [6]. Let me remark that

$$\sigma_c(f^k) \geq 1 - \frac{1}{k}, \quad k = 2, 3, \dots,$$

a converse of an extended Stieltjes theorem (see appendix).

The conclusion of Bohr is that σ_c is not very well connected with intrinsic properties of $f(s)$, at least not with the order function $\mu(\sigma)$. Henry Helson reconsidered the question in 1962 and gave a very elegant fomula for σ_c using Fourier properties of $f(s)/s$ considered as a function of t $(s = \sigma + it)$ [11].

B: The summability theory

Given an ordinary Dirichlet series (1), let us write now

$$\lambda_0 = \sigma_c$$

$$\lambda_r = \text{abscissa of } C^r\text{-summability of (1)}$$

where C^r is the Cesàro process of summation of order r. In 1909, Bohr considers only integral values of r; in the 1950's, following M. Riesz, general $r > 0$. The "summability function" is $\psi(\sigma)$ defined by

$$\psi(\lambda_r) = r,$$

that is, (1) is C^r-summable at $s = \sigma + it$ if $r < \psi(\sigma)$ and is not C^r-summable at any $s = \sigma + it$ such that $r > \psi(\sigma)$. Bohr's theory leads to

$$\psi(\sigma) \leqslant \mu(\sigma) \leqslant \psi(\sigma) + 1 \tag{4}$$

together with

$$\begin{cases} \psi \text{ convex and } \psi(\sigma) = 0 \text{ for large } \sigma \\ \psi'(\sigma - 0) \leqslant -1 \text{ or else } \psi(\sigma) = 0 \end{cases} \tag{5}$$

(figure 3). As a consequence, the half plane where (1) is C^r-summable for some $r > 0$ and the maximal half plane where the function $f(s)$ represented by (1) is holomorphic and of bounded order are the same (up to the boundary), a striking and final result – actually, the best result of his thesis –.

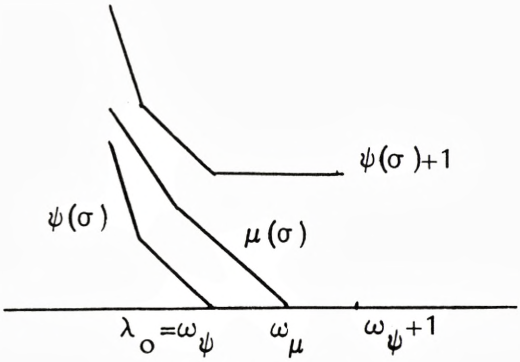

Fig. 3

However, Bohr was not satisfied. Given two functions $\psi(.)$ and $\mu(.)$ as in figure 3, is it possible to construct an ordinary Dirichlet series having them as summability and order function respectively? In his last paper [7] Bohr solves the question completely as far as $\psi(.)$ is concerned: (5) is necessary and sufficient for $\psi(.)$ to be a summability function. What about $\mu(.)$? Assuming (4) and the analogue of (5) for $\mu(.)$, that is $\mu(.)$ is convex and

$$\mu' \, (\omega_\mu - 0) \leqslant -1 \tag{6}$$

(where $\omega_\mu = \inf \{\sigma\,;\ \mu(\sigma) = 0\}$), then Bohr shows that $\{\psi(.), \mu(.)\}$ is actually a couple {summability function, order function}.

Now, is (6) a necessary condition (when the first member exists)? This is the last problem of Harald Bohr [7].

Here is a previous problem [4]. Does there exist a Dirichlet series (1) with $\sigma_c = 0$, $\sigma_a = 1$, $\mu(\sigma) = \sup(0, \frac{1}{2} - \sigma)$ (figure 4)?

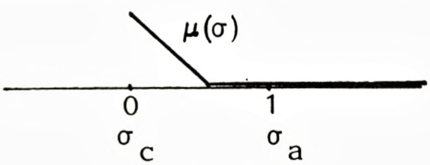

Fig. 4

Both questions are inspired by the Riemann ζ-function. If (6) were a necessary condition, it would prove the Lindelöf hypothesis for $\zeta(s)$, that is

$$\zeta(\frac{1}{2} + it) = O(|t|^{\epsilon}) \ (t \to \infty)$$

for all $\epsilon > 0$. If the Lindelöf hypothesis is true, $\sum_1^\infty (-1)^n n^{-s}$ provides a positive answer to the second question.

3. *After 1951 (a personal selection)*

I already mentioned Helson's formula for σ_c. Following the same idea – Fourier methods in Dirichlet series – Helson gave a very elegant proof of the prime number theorem [12].

Playing with $\mp$ in series (2) gives interesting problems and results. I introduced the game in 1974 and it was developed by H. Queffelec [15] [21]. The first interesting example is

$$\sum_1^\infty \epsilon_n ((2n-1)^s - (2n)^s) \tag{7}$$

with $\epsilon = (\epsilon_1, \epsilon_2, \ldots) \in \{-1, 1\}^\infty = \Omega$. If we consider Ω as a probability space with the natural probability, figure 4 holds almost surely, which solves the second-mentioned Bohr problem. If we consider Ω as a topological space, then figure 2 holds quasi-surely (meaning: on a dense G_δ-set), which replaces a rather technical construction in Bohr's thesis.

Instead of differences of the first order in (7) it is possible to consider differences of higher and higher order, and get Dirichlet series for which $\mu(\sigma) = \sup(0, \frac{1}{2} - \sigma)$ on $(-\infty,\infty)$ (almost surely) or $\mu(\sigma) = \sup(0, 1-\sigma)$ on $(-\infty,\infty)$ (quasi-surely). That helps in constructing the "building blocks" from which Bohr's theorem on $\{\psi(.), \mu(.)\}$ derives (see [21] and [16]).

Quite different results are obtained by Queffelec [21] for almost sure and quasi sure properties of Euler products

$$\prod_1^\infty (1 + \epsilon_n n^{-s})$$

General random Dirichlet series

$$\sum_1^\infty a_n(\omega)\, e^{-\lambda_n s}$$

and their growth properties are considered by Yu Jia-rong [30].

The Harald Bohr centenary was a good opportunity to investigate the last problem, on the order functions $\mu(\sigma)$ of ordinary Dirichlet series. Here are the results I obtained:

1) a necessary condition is $\mu(\sigma + \mu(\sigma) + \frac{1}{2}) = 0$.

2) (6) is not necessary. It is possible to have $\mu'(\omega_\mu - 0)$ as near $\frac{1}{2}$ as one wants [16].

Therefore, the last problem splits into two parts:

1) find another approach to the Lindelöf hypothesis,

2) characterize the order functions of ordinary Dirichlet series. For example, for which $\alpha \geq 0$ can one have

$$\mu(\sigma) = \sup(0, \frac{1}{2} - \sigma, \alpha(1-\sigma))?$$

($\alpha \leqslant \frac{1}{2}$ is necessary, $\alpha = 0$ is sufficient).

4. *Appendix*

1. Here is the proof of Stieltjes's theorem on mulitplication of Dirichlet series. We consider $\Sigma\, a_n n^{-s}$, $\Sigma\, b_n n^{-s}$ and their product $\Sigma\, c_n n^{-s}$. Assume that $\Sigma\, a_n$ and $\Sigma\, b_n$ converge. Given N,

$$\sum_1^N c_n = \sum_{(m,p):mp\leqslant N} a_m b_p$$

$$= \sum_{1\leqslant m\leqslant\sqrt{N}} \Big(a_m \sum_{1\leqslant p\leqslant N/m} b_p\Big) + \sum_{1\leqslant p\leqslant\sqrt{N}} \Big(b_p \sum_{\sqrt{N}<m\leqslant N/p} a_m\Big)$$

$$= O(\sqrt{N}),$$

hence $\Sigma\, c_n n^{-\sigma}$ converges for $\sigma > \frac{1}{2}$, QED.

In the same way, given k series $\Sigma\, a_n^{(j)} n^{-s}$ $(j = 1, 2, \ldots k)$ which converge for $s = 0$, their product is a Dirichlet series which converges for $\sigma > 1 - \frac{1}{k}$.

2. I mentioned the beautiful arguments of Jensen and Bohr proving that $(1-s)\zeta(s)$ is an entire function. However the classical proof is the Rieman functional equation. Here is a simple way to express the proof of the functional equation. Let $E(x)$ = integral part of x for $x \geq 0$, $E(-x) = E(x)$. Then

$$\zeta(s) = \int_0^\infty x^{-s}\, d(E(x) - x)$$

for $0 < \mathrm{Re}\, s < 1$ and

$$d(E(x) - x) = \int_0^\infty (e^{2\pi itx} + e^{-2\pi itx})\, d(E(t) - t)$$

in the sense of Schwartz. Through a simple regularisation (multiplying x^{-s} by a C^{∞}-function with compact support in $]0, \infty[$) we have

$$\zeta(s) = \int_0^{\infty} \left(\int_0^{\infty} x^{-s} \, (e^{2\pi itx} + e^{-2\pi itx}) dx \right) d(E(t)-t)$$
$$= C(s) \int_0^{\infty} t^{s-1} \, d(E(t)-t) = C(s) \; \zeta(1-s)$$

with $C(s) = 2 \int_0^{\infty} x^{-s} \cos 2\pi x \, dx$.

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Université de Paris-Sud
Mathématiques-Bât. 475
91405 Orsay Cedex
France

Bohr Almost Periodicity and Functions of Dynamical Type

By Paul Milnes

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In this note we define the space $BAP(G)$ of Bohr almost periodic functions on a locally compact group G and, after reviewing the basic implications of the definition, discuss examples of functions that are Bohr almost periodic, but not almost periodic in the sense of Bochner. These examples are either due to or inspired by T.-S. Wu. We then consider dynamical properties of $BAP(G)$, showing among other things that $BAP(G) \subset MIN(G)$, the space of minimal functions on G. We also mention some pathologies; for example, $BAP(G)$ need not be a linear space. A concluding result, which we quote, is due to A. L. T. Paterson and may be thought of as a regularity property of $BAP(G)$. It asserts that $BAP(G)$ consists of left almost convergent functions.

A way to view one aspect of Harald Bohr's achievement with his theory of almost periodic functions is that he provided a characterization of the norm closed, linear span of the continuous periodic functions on $\mathbf{R}$. It is clear that any attempt to characterize this space must overcome the apparent problem that even the sum of two periodic functions is usually not periodic, e.g., $x \to \sin x + \sin(\sqrt{2}x)$. Earlier attempts at some aspects of such a characterization had been made by Bohl [3] and Esclangon [6].

Bohr defined a continuous complex-valued function f on $\mathbf{R}$ to be almost periodic if: for any $\epsilon > 0$, there is a real number $L(\epsilon) > 0$ such that every interval of length $L(\epsilon)$ contains at least one translation number of f corresponding to ϵ. (See Bohr [4; pp. 31-2], for example. [5] has an extensive bibliography.) We write this in symbols:

for any $\epsilon > 0$, there is a real number $L_\epsilon > 0$ such that $[r, r + L_\epsilon] \cap \{s \mid |f(t+s) - f(t))| < \epsilon$ for all $t \in \mathbf{R}\} \neq \emptyset$ $(r \in \mathbf{R})$. The rationale for the term "almost periodic" is obvious; if the real number $L_\epsilon > 0$ exists for $\epsilon = 0$, then f is periodic. Also obvious is how to generalize the setting.

Definition 1. A continuous complex-valued function f on a locally compact group G is called *Bohr almost periodic* if:

for any $\epsilon > 0$ there is a compact $K_\epsilon \subset G$ such that

$$(rK_\epsilon) \cap \{s \in G \mid |f(ts) - f(t)| < \epsilon \text{ for all } t \in G\} \neq \emptyset \ (r \in G). \tag{1}$$

Let $BAP(G)$ denote the class of Bohr almost periodic functions on G.

Although the definition does not require $f \in BAP(G)$ to be bounded, it does require

$$\|R_s f - f\| := \sup_{t \in G} |(R_s f - f)(t)| = \sup_{t \in G} |f(ts) - f(t)| < \epsilon$$

for many $s \in G$. Also, since $\{s \mid \| R_s f - f\| < \epsilon\}$ is a symmetric set, (1) is equivalent to

$$K_\epsilon \{s \in G \mid \|R_s f - f\| < \epsilon\} = G, \tag{1'}$$

and to

$$\text{for each } t \in G, \text{ there is a } k \in K_\epsilon \text{ such that } \|R_t f - R_k f\| < \epsilon. \tag{1''}$$

Since a function in $BAP\ (S)$ must in fact be bounded, the formulation (1'') shows that the compact sets K_ϵ can always be chosen finite if and only if the *orbit* $R_G f := \{R_s f \mid s \in G\}$ is totally bounded, i.e., f is almost periodic in the sense of Bochner [2]. Thus, denoting by $AP(G)$ the class of Bochner almost periodic functions, we note that $BAP(G) = AP(G)$ if G is discrete.

2. Here are some facts about $BAP(G)$. Their demonstration can usually be modelled on proofs in Bohr [4]; see also [8, 1]. ([10] is a standard reference for topological groups.)

(a) The functions in $BAP(G)$ are bounded (as mentioned above) and right uniformly continuous. (We write $BAP(G) \subset \mathscr{U}_r(G)$; a function $f\colon G \to \mathbf{C}$ is *right uniformly continuous* if, for all $\epsilon > 0$, there is a neighbourhood V of the identity $e \in G$ such that $|f(s) - f(t)| < \epsilon$ whenever $st^{-1} \in V$.)

(b) $BAP(G)$ is norm closed in $\mathscr{U}_r(G)$ and *translation invariant* (i.e., $f \in BAP(G)$ and $s \in G$ imply $R_s f$, $L_s f \in BAP(G)$, where $L_s f(t) = f(st)$).

(c) $BAP(G) \cap \mathscr{U}_l(G) = AP(G)$. (Here $\mathscr{U}_l(G)$ is the analogously defined space of bounded functions that are *left* uniformly continuous.)

From (a) and (c) it follows that $BAP(G) = AP(G)$ if $\mathscr{U}_r(G) = \mathscr{U}_l(G)$ (for example, if G is abelian). The converse is an open question. A group to look at in this connection is the affine group of the line $\mathbf{R} \otimes \mathbf{R}^+$, for which we suspect $BAP = AP$, although $\mathscr{U}_r \neq \mathscr{U}_l$.

The definition of $BAP(G)$ uses right translates. We denote by $LBAP(G)$ the space defined analogously using left translates. It follows that $LBAP \subset \mathscr{U}_l$, that $LBAP \cap BAP = AP$, and that the equality $LBAP = BAP$ implies the identity of all three spaces, $LBAP = BAP = AP$.

Examples 3. The examples presented here of functions in $BAP \backslash AP$ are due to or inspired by Wu [17]. A more detailed treatment of them can be found in [12, 13, 14].

(i) On $G = \mathbf{C} \otimes \mathbf{T}$, the euclidean group of the plane with multiplication $(z', w')\,(z, w) = (z' + w'z, w'w)$, the function $f(z, w) = e^{i\mathrm{Re}(z/w)}$ is in $BAP \setminus \mathscr{U}_l$, as is readily verified. (Here Re indicates real part.)
(ii) On $G = (\mathbf{T} \times \mathbf{T}) \otimes \mathbf{Z}$ with multiplication $(w_1', w_2', n')\,(w_1, w_2, n) = (w_1' w_1 w_2^{n'}, w_2' w_2, n' + n)$, the function $f(w_1, w_2, n) = w_1$ satisfies $R_{(1,1,m)} f = f$ for all $m \in \mathbf{Z}$. Hence $f \in BAP(G)$, since we can choose $K_\epsilon = \mathbf{T} \times \mathbf{T} \times \{0\}$ for all $\epsilon > 0$ in Definition 1. However $f \notin \mathscr{U}_l$. (This is Wu's method [17] and works more generally: if $G = G_1 \otimes G_2$ is a semidirect product with G_1 compact, and if $F \in C(G_1)$, then $f(s, t) = F(s)$ defines an $f \in BAP(G)$.)
(iii) Let $G = \mathbf{T}^{\mathbf{T}} \otimes \mathbf{T}_d$, where $\mathbf{T}^{\mathbf{T}}$ is the compact group of all functions from $\mathbf{T}$ into $\mathbf{T}$ and $\mathbf{T}_d$ is the discrete circle group. The product in G is $(h', w')\,(h, w) = (h' R_{w'} h, w'w)$. Let $f(h, w) = h(1)$. Then $f \in BAP \setminus \mathscr{U}_l$. Further, define $g \in \mathbf{T}^{\mathbf{T}}$ by $g(-1) = -1$, $g(w) = 1$ otherwise. Then, by 2(b), $R_{(g,1)} f \in BAP$. However $f + R_{(g,1)} f \notin BAP$.

We now want to make a connection with topological dynamics. If $f \in \mathscr{U}_r(G)$, then the closure $X_f := \overline{R_G f}$ of the orbit $R_G f$ in the topology of pointwise convergence on G is compact in $\mathscr{U}_r(G)$ for that topology. The translation operators $R_t, t \in G$, leave X_f invariant and (R_G, X_f) is a flow. f is called *minimal, point distal* or *distal* if that flow is minimal, point distal with f as distal point, or distal, respectively. Specifically, an $f \in \mathscr{U}_r(G)$ is:

minimal if, whenever $h_1 = \lim_\alpha R_{s_\alpha} f$ (pointwise on G), there is a net $\{t_\beta\} \subset G$ such that $f = \lim_\beta R_{t_\beta} h$;

point distal if, whenever $h_1 = \lim_\alpha R_{s_\alpha} f$ and $\lim_\beta R_{t_\beta} h_1 = h' = \lim_\beta R_{t_\beta} f$, it follows necessarily that $h_1 = f$;

or

distal if, whenever $h_1 = \lim_\alpha R_{s_\alpha} f$, $h_2 = \lim_\beta R_{t_\beta} f$ and $\lim_\gamma R_{r_\gamma} h_1 = h' = \lim_\gamma R_{r_\gamma} h_2$, it follows necessarily that $h_1 = h_2$.

We denote the classes of minimal, point distal and distal functions on G by $MIN(G)$, $PD(G)$ and $D(G)$, respectively. Clearly distal functions are point distal, and point distal functions are minimal [7, 11, 1]. The functions in Examples 3, (i) and (ii), are distal, but the one in (iii) is in $MIN(G) \setminus PD(G)$. (A function f that is in $PD(\mathbf{Z}) \setminus (D(\mathbf{Z}) \cup BAP(\mathbf{Z}))$ is defined by $f(n) = \cos n / |\cos n|$.)

We quote two theorems.

THEOREM 4 [7,11]. Let $f \in \mathscr{U}_r(G)$. Then $f \in MIN(G)$ if and only if:

$$(*) \qquad \begin{cases} \text{for all } \epsilon > 0 \text{ and finite } F \subset G, \text{ there is a finite} \\ K_{\epsilon,F} \subset G \text{ such that} \\ K_{\epsilon,F}\{s \in G \mid |f(ts) - f(t)| < \epsilon \text{ for all } t \in F\} = G. \end{cases}$$

THEOREM 5 [14]. $BAP(G) \subset MIN(G)$.

The condition (*) in Theorem 4 looks similar to the (1') formulation of the definition of Bohr almost periodicity; indeed, one can show directly that a Bohr almost periodic function satisfies (*). The proof of Theorem 5 given in [14] shows that, if $f \in BAP(G)$, $h \in X_f$ and $\epsilon > 0$, then there is a $t \in G$ such that

$$(**) \qquad \|R_t h - f\| \leq \epsilon$$

(which proves $f \in MIN(G)$).

Remarks 6. (i) (**) is equivalent to $\| h - R_{t^{-1}} f \| \leq \epsilon$, from which we conclude that, for an $f \in BAP(G)$, X_f, which is the pointwise closure of $R_G f$, equals the norm closure of $R_G f$; we write

$$R_G f^{-p} = R_G f^{-\|\,\|}. \tag{1}$$

It follows from 2(b) that $X_f \subset BAP(G)$.

(ii) Clearly an $f \in \mathscr{U}_r(G)$ that satisfies (1) is in $MIN(G)$. However, not all minimal functions f satisfy (1). A class of minimal functions that do not satisfy (1) is $PD(G) \setminus D(G)$, hence $BAP \cap PD = BAP \cap D$.

(iii) Suppose an $f \in \mathscr{U}_r$ satisfies (1). Does this always imply $f \in BAP$? Not without some connectivity hypothesis. For, suppose $f \in BAP \setminus AP$ on some group G. Then some of the K_ϵ's in Definition 1 cannot be chosen finite, hence f is not Bohr almost periodic on the discrete group G_d. But (1) still holds for f on G_d.

In Example 3 (iii) we pointed out that $BAP(G)$ need not form a linear space. Here are two more unusual aspects of $BAP(G)$.

(a) If G satisfies $BAP \setminus AP \neq \emptyset$, consider the identity map $\imath : G_d \to G$. Although $\imath$ is a continuous homomorphism, the adjoint map $\imath^*$, $\imath^*(f) := f \circ \imath$, does not map $BAP(G)$ into $BAP(G_d)$. (Of course, $\imath^*(AP(G)) \subset AP(G_d)$, etc.)

(b) Let G and f be as in Example 3 (iii). Then *every* $\mathbf{T}$-valued function h on the subgroup $\{1\} \times \mathbf{T}_d$ extends to a function $R_{(h,1)} f \in BAP(G)$. (Note that, if H_1 is a subgroup of a group H and $f \in AP(H)$, for example, then the restriction of f to H_1 is in $AP(H_1)$.)

We quote two more theorems.

THEOREM 7 [14]. Let ψ be a continuous open homomorphism of G_1 onto G_2. Then $\psi^*(BAP(G_2)) \subset BAP(G_1)$.

THEOREM 8 (A. L. T. Paterson). Let G be an amenable locally compact group. Then each $f \in BAP(G)$ is left almost convergent.

We refer the reader to [9, 15, 16] for amenability. A function $f \in \mathscr{U}_r(G)$ is *left almost convergent* if the set

$$\{\mu(f) \mid \mu \text{ is left invariant mean on } \mathscr{U}_r(G)\}$$

is a singleton. Paterson proved Theorem 8 by showing that an $f \in BAP(G)$ has a constant function in its norm closed convex hull.

In conclusion we remark that one can consider Bohr almost periodic functions on topological groups that are not locally compact. All the results here go through unchanged in this more general setting. One can even extend the setting to semi-topological groups; in this setting an $f \in BAP(G)$ is defined to satisfy the condition of Definition 1 and also to be in $\mathscr{U}_r(G)$.

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University of Western Ontario
London, Ontario, N6A 5B7
Canada

Almost-Periodic Solutions of Navier-Stokes Equations and Inequalities

By Giovanni Prouse

Introduction

In this talk I would like to present some results, old and new, concerning almost-periodic solutions of Navier-Stokes equations and inequalities, which govern the motion of viscous compressible or incompressible fluids (respectively gases or liquids).

Of the various problems which can be associated with this motion I shall, in what follows, for the sake of simplicity, consider only the one corresponding to a fluid in a bounded 2- or 3-dimensional domain Ω, which boundary Γ constituted by a material surface. Denoting by $\vec{u}\ (x,t)$ $(x=\{x_1, x_2, x_3\} \in \overline{\Omega})$ the velocity of the fluid, the problem indicated above corresponds, by the limit layer theory, to the homogeneous Dirichlet boundary condition

$$\vec{u}\ (x,t) = 0 \qquad\qquad (x \in \Gamma). \tag{1.1}$$

The following notations will be used in the sequel.

$\vec{f}(x,t)$ external force acting on the fluid;
$p(x,t)$ pressure;
$\varrho(x,t)$ density; in the incompressible case (ϱ = const) I shall assume, for simplicity, $\varrho = 1$;
μ, ζ viscosity coefficients (resp. shear and bulk viscosity);
$\mathscr{D}$ space of functions (or vectors) $\in C^\infty(\overline{\Omega})$ and with compact support in Ω;
$\mathscr{N}$ space of vectors $\vec{v} \in \mathscr{D}$ and such that div $\vec{v} = 0$;
H^s (s integer $\geqslant 0$) space of functions (or vectors) square summable in Ω, together with their derivatives (in the sense of distributions) of order $\leqslant s$;
N^s closure of $\mathscr{N}$ in H^s.

The most common mathematical model associated to the motion of a viscous fluid is constituted by the *Navier-Stokes equations* which, in the case of incompressible fluids, take the form

$$\begin{cases} \dfrac{\partial \vec{u}}{\partial t} - \mu\Delta\vec{u} + (\vec{u}.\text{grad})\ \vec{u} + \text{grad}\ p = \vec{f} \\ \\ \text{div}\ \vec{u} = 0. \end{cases} \tag{1.2}$$

while, if the fluid is compressible, are expressed by

$$\begin{cases} \varrho \dfrac{\partial \vec{u}}{\partial t} + (\zeta + \dfrac{1}{3}\mu) \text{ grad div } \vec{u} - \mu \Delta \vec{u} + \varrho\, (\vec{u}.\text{grad})\, \vec{u} + \text{grad } p = \varrho \vec{f} \\ \dfrac{\partial \varrho}{\partial t} + \text{div } (\varrho \vec{u}) = 0 \\ p = p(\varrho). \end{cases} \tag{1.3}$$

The third equation of (1.3) is an *equation of state* which, in most practical cases, is given by $p = k\varrho^{\gamma}$ $(k, \gamma > 0)$.

It should be noted that (1.2) cannot be considered as a special case of (1.3), since the two systems are essentially different.

Another model associated to viscous incompressible flow corresponds to the *Navier-Stokes inequalities* which are introduced as follows. Observing that the Navier-Stokes equations are non-relativistic and, consequently, do not have any physical meaning when $|\vec{u}|$ approaches the velocity of light, the model (1.2) is equivalent, from a physical point of view, to the one corresponding to the relationships

$$\begin{cases} \dfrac{\partial \vec{u}}{\partial t} - \mu \Delta \vec{u} + (\vec{u}.\text{grad})\, \vec{u} + \text{grad } p = \vec{f} \quad \text{where } |\vec{u}| < c \\ \text{div } \vec{u} = 0 \quad , \quad |\vec{u}| \leqslant c \\ \vec{u} \text{ continuous at the “interfaces” of the two sets in which resp. } |\vec{u}| < c \text{ and } |\vec{u}| = c. \end{cases} \tag{1.4}$$

It is well known, on the other hand, from the theory of differential inequalities (see, for instance, Lions [1]) that (1.4) is equivalent to the system

$$\begin{cases} \displaystyle\int_{t_1}^{t_2} \int_{\Omega} \left(\frac{\partial \vec{u}}{\partial t} - \mu \Delta \vec{u} + (\vec{u}.\text{grad})\, \vec{u} + \text{grad } p - \vec{f} \right) (\vec{u} - \vec{\varphi})\, dt\, d\Omega \leqslant 0. \\ \text{div } \vec{u} = 0 \\ |\vec{u}| \leqslant c \end{cases} \tag{1.5}$$

$\forall\, \vec{\varphi}$ such that $|\vec{\varphi}| \leqslant c$ and $\forall\, t_1, t_2 \in (-\infty, +\infty)$.

System (1.5) therefore constitutes an *inequality model* for the problem considered, in the incompressible case. An analogous model could obviously be given for compressible fluids, but it will not be considered here.

In the next section I shall recall some results concerning the almost-periodic solutions of the three models presented; it is however useful to first briefly summarize the main existence and uniqueness theorems of a solution of (1.2), (1.3), (1.5) satisfying (1.1) and the initial conditions

$$\vec{u}(x, 0) = \vec{u}_0(x) \qquad \text{(incompressible case)}$$
$$\vec{u}(x, 0) = \vec{u}_0(x)\, ,\ \varrho(x, 0) = \varrho_0(x) \qquad \text{(compressible case)}$$

These theorems represent, in fact, the first step in the study of the almost-periodic solutions.

The solutions will always be intended in the sense of distributions, while I shall not, for the sake of simplicity, indicated explicitly the functional spaces in which the solutions are found, or the assumptions on the data.

Considering system (1.2), Hopf [2] proved the global (in time) existence of a solution in any space dimension; the uniqueness of such a solution can however be guaranteed only in 2 dimensions (Lions and Prodi [3]). An existence and uniqueness theorem in $\Omega \times (0, T)$, Ω 3-dimensional, holds provided $\vec{f}$ is "sufficiently small" (Kieselev and Ladyzenskaja [4]).

One can, on the other hand, prove a global existence and uniqueness theorem for the solution in $\Omega \times (0, T)$ of (1.5) (Prouse [5]).

In the compressible case, only a local existence and uniqueness theorem holds (Valli [6]); in order to obtain global existence and uniqueness, one must assume that $\vec{f}$ is "sufficiently small" (Marcati and Valli [7]).

Almost-periodicity theorems

The models introduced in the preceding section all correspond to dissipative problems, and the study of their almost-periodic solutions follows therefore from the guidelines given, for ordinary dissipative differential equations, by Favard [8] and Amerio [9] respectively in the linear and non linear case.

In the theory of almost-periodic solutions of partial differential equations, vector valued functions play an essential role, together with the concepts of weakly almost-periodic and S^p-Stepanov almost-periodic functions. For these concepts and for the basic definitions and properties of functions with values in a Banach space, I refer to the note by L. Amerio which appears in the present volume (see also Amerio, Prouse [10]).

While the details of the proofs of the existence and uniqueness of an almost-periodic solution, under the assumption that $\vec{f}(t)$ is almost-periodic, are obviously different for the three models considered, the basic scheme is similar and consists essentially of the following steps:

a) A global existence theorem in $[t_0, +\infty)$;

b) An existence and uniqueness theorem of a solution $\vec{u}(t)$ (or $\{\vec{\tilde{u}}(t), \tilde{\varrho}(t)\}$) bounded on $J = (-\infty, +\infty)$ (assuming $\vec{f}(t)$ bounded on J);

c) The proof that $\vec{u}(t)$ ($\{\vec{\tilde{u}}(t), \tilde{\varrho}(t)\}$) is weakly almost-periodic if $\vec{f}(t)$ is weakly almost-periodic;

d) The proof that the range of $\vec{u}(t)$ ($\{\vec{\tilde{u}}(t), \tilde{\varrho}(t)\}$) is relatively compact if $\vec{f}(t)$ is almost-periodic.

Observe that point a) corresponds essentially to the results recalled in the preceding section, setting $T = +\infty$.

Assuming that $f(t)$ is S^2-Stepanov almost-periodic, the following theorems then hold.

THEOREM I (Prouse [11]): *If Ω is 2-dimensional, $\vec{f}(t) \in L^\infty(J; L^2)$ and is "sufficiently small",* (1.1), (1.2) *admit a unique solution $\tilde{\vec{u}}(t)$ which is N^0 -Bohr and N^1-S^2-Stepanov almost-periodic.*

THEOREM II (Foias [12], Heywood [13]): *If Ω is 3-dimensional and of class C^3, $\vec{f}(t) \in L^2_{\text{loc}}(J, N^1) \cap H^1_{\text{loc}}(J, (N^1)^*)$ and is "sufficiently small", then* (1.1), (1.2) *admit a unique solution $\tilde{\vec{u}}(t)$ which is N^0-Bohr and N^1-S^2-Stepanov almost-periodic.*

THEOREM III (Marcati and Valli [7]): *If Ω is 3-dimensional and of class C^4, $p \in C^3$, $p' > 0$, $\vec{f}(t) \in L^2_{\text{loc}}(J; H^1) \cap H^1_{\text{loc}}(J; H^{-1})$ and is "sufficiently small", then* (1.1), (1.3) *admit a unique solution $\{\tilde{\vec{u}}(t), \tilde{\varrho}(t)\}$ with $\tilde{\vec{u}}(t)$ H^1-Bohr and H^2-S^2-Stepanov almost-periodic, $\tilde{\varrho}(t)$ L^2-Bohr and H^2-S^2-Stepanov almost-periodic.*

THEOREM IV (Prouse [14]): *If Ω is 3-dimensional, $\vec{f}(t) \in L^\infty(J; L^2)$ and is "sufficiently small", then* (1.1), (1.4) *admit a unique solution $\tilde{\vec{u}}(t)$ which is N^0-Bohr and N^1-S^2-Stepanov almost-periodic.*

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Politecnico di Milano
Dipartimento di Matematica
20133 Milano, Piazza Leonardo da Vinci 32
Italy

On the Zeros of Entire Almost Periodic Functions

By HANS TORNEHAVE

We shal prove in this paper that a lattice $\Omega \subset \mathbf{C}$ which is the set of zeros of an entire almost periodic function $f\colon \mathbf{C} \to \mathbf{C}$ is periodic in the direction of almost periodicity.

A method for construction of holomorphic almost periodic functions was introduced in [4] and was applied more extensively in [6]. During the work on [6] the authors noticed that the method failed if the set of zeros was a lattice and not periodic in the direction of almost periodicity. The authors discussed it only briefly and it was not mentioned in the paper.

A rotation of both the lattice and the direction of almost periodicity around the point 0 and by the same angle will have no influence on the existence of almost periodic entire functions with the given lattice as set of zeros.

Accordingly, we shall assume that the given direction of almost periodicity is the direction of the real axis and that the lattice Ω is not periodic in this direction, i.e. that 0 is the only real number in Ω. We shall study a hypothetical entire almost periodic function $f\colon \mathbf{C} \to \mathbf{C}$ with $f^{-1}(0) = \Omega$. We are going to prove the non existence of such a function by deducing that some function derived from f will have properties which contradict each other.

The 8 lemmas of this paper are statements directly or indirectly dealing with the non existing function f. Hence, they have no applications whatever beyond the scope of this paper. The 7 propositions are genuine statements about rather general classes of functions, but most of them are reformulations of known results adopted for our particular purpose.

The first section states the problem, introduces some notions and does some preliminary work. It ends with the key lemmas 2 and 3, which state that f and some related functions cannot assume very small values except near the zeros.

The second section investigates the Fourier series of f. It turns out that the 2-dimensionality of the lattice of zeros is reflected in the set of Fourier exponents. In fact the subspace of the $\mathbf{Q}$-vector space $\mathbf{R}$ generated by the set of Fourier exponents has a 'compulsory' 2-dimensional subspace determined by Ω.

In the third section we introduce the spatial extension of f, i.e. a function $F\colon \mathbf{R}^\infty \times \mathbf{R} \to \mathbf{C}$ with $f(z) = F(\gamma x; y)$; $z = x + iy$. Here, F is limit periodic and $\gamma = (\gamma_1, \gamma_2, \ldots)$ is a base for the vector space generated by the Fourier exponents such that (γ_1, γ_2) spans the compulsory subspace. If M denotes the zeros of F in the $(x_1, x_2; y)$-subspace of $\mathbf{R}^\infty \times \mathbf{R}$, we have $F^{-1}(0) = p^{-1}(M)$ when $p\colon \mathbf{R}^\infty \times \mathbf{R} \to \mathbf{R}^2 \times \mathbf{R}$ is the projection. Further, M is a system of parallel straight lines, each connecting a point of Ω placed in $\mathbf{R}^\infty \times \mathbf{R}$ by $f(z) = F(\gamma x; y)$ and projected on the subspace, with a point of the unit lattice in the

(x_1, x_2)-plane. The proof of this is the tiresome part of the paper and the author hopes that somebody will find a more elegant way of doing it.

The fourth section finishes the proof of the non existence of f by a topological argument. We know that f has the variation of its argument around each zero equal to 2π. It is possible to let small circles around the zeros of f slide along the lines of M to end in the (x_1, x_2)-plane and this enables us to prove that also the restriction $\varphi(x_1, x_2) = F(x_1, x_2, 0, 0, \ldots, 0)$ by convenient orientation of the (x_1, x_2)-plane has the variation of the argument around each zero equal to 2π. The lemmas 2 and 3 will also carry over and that makes it possible to prove that the variation of the argument of F along the boundaries of certain big squares has to be zero and also to be a very large number and that is the contradiction.

In the fifth and last section we shall prove that there is a lattice $\Omega' \subset \mathbf{C}$ and a second order entire almost periodic function with $\Omega \cup \Omega'$ as its set of zeros.

Almost periodic properties of the function f

The field $\mathbf{R}$ of real numbers is also a $\mathbf{Q}$-vector space and we shall use the notion of linear independence accordingly. If (x_j) is a sequence of real numbers which are linearly independent over $\mathbf{Q}$ we shall simply say that the numbers x_j, the sequence (x_j) or the set $\underline{x} = (x_1, x_2, \ldots)$ are independent.

We shall assume that the lattice Ω is spanned by the complex numbers $\omega_1 = \alpha_1 + i\beta_1$, $\omega_2 = \alpha_2 + i\beta_2$, $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbf{R}$, and that the indices are chosen such that $\alpha_1\beta_2 - \alpha_2\beta_1 = \Delta > 0$. We shall also assume that Ω is not periodic in the direction of the real axis, and this is equivalent to the assumption that $\Omega \cap \mathbf{R} = \{0\}$ and also to the assumption that β_1 and β_2 are independent.

We shall call a set $T \subseteq \mathbf{C}$ relatively dense if there exists a real number L, such that every closed interval $I \subset \mathbf{R}$ of length L contains the real part of at least one element of T. By Kronecker's theorem and the Bohl-Wennberg theorem ([6] p. 145, footnote) the following statement holds:

For every $\delta > 0$ and every $y \in \mathbf{R}$ the set of numbers $\omega \in \Omega$ with imaginary part in the interval $[y - \delta, y + \delta]$ is relatively dense.

We shall consistently use z with or without indices as notation for a complex number, and always with $z = x + iy$ and the indices repeated on x and y. To a bounded interval $I \subset \mathbf{R}$ corresponds a strip $S_I = \{z \mid y \in I\}$. A strip $S \subset \mathbf{C}$ is a set defined in this way. We shall write I_S for the interval defining S. Mostly, we have $I = [-A, A]$ with some $A > 0$ and we shall then write S_A for S_I.

Most of the functions considered here will be continuous functions g: $\mathbf{C} \to \mathbf{C}$, but not always holomorphic. We define the absolute value $|g|$ by $|g|(z) = |g(z)|$. We shall permit ourselves the abuse of notation of confusing a function with its value, e.g. by writing "the function $g(z)e^{\gamma z}$" meaning "the function $\tilde{g}$: $\mathbf{C} \to \mathbf{C}$ defined by $\tilde{g}(z) = g(z)e^{\gamma z}$". For $\tau \in \mathbf{C}$ we use the notation g_τ: $\mathbf{C} \to \mathbf{C}$ for the translated function $g_\tau(z) = g(z+\tau)$. For $\varepsilon > 0, A > 0$ we call $\tau \in \mathbf{C}$ an (ε, A)-translation number of g if $|g_\tau(z) - g(z)| \leqq \varepsilon$ for every $z \in S_A$. According to Bohr's definition g is almost periodic if g is continuous and the set of real (ε, A)-translation numbers is relatively dense for every $\varepsilon > 0, A > 0$. This definition is equivalent to Bochner's definition, according to which g: $\mathbf{C} \to \mathbf{C}$ is almost periodic if g is continuous and every sequence $(\tau_j | j \in \mathbf{N})$ of real numbers has a subsequence (τ'_j) such that the sequence $(g_{\tau'_j})$ converges uniformly in every strip. This can be generalized in the following way:

Proposition 1. *Let g: $\mathbf{C} \to \mathbf{C}$ be almost periodic and let S be a strip. Then every sequence (τ_j) of complex numbers $\tau_j \in S$ has a subsequence (τ'_j) such that $(g_{\tau'_j})$ converges uniformly in every strip.*

Proof: We write $\tau_j = \rho_j + i\sigma_j$ and we can then choose the subsequence (τ'_j) with $\tau'_j = \rho'_j + i\sigma'_j$ such that $(g_{\rho'_j})$ converges uniformly in every strip and (σ'_j) converges to a limit σ. Then, obviously $(g_{\rho'_j+i\sigma})$ converges uniformly in every strip, and since g is uniformly continuous in every strip, the sequence $(g_{\tau'_j} - g_{\rho'_j+i\sigma})$ tends to 0 uniformly in every strip, and the statement follows.

We shall use the following statement, which is pure function theory and not very exciting:

Proposition 2. *Let $\mathscr{B}$ denote the $\mathbf{C}$-vector space of entire functions bounded in every strip with the Fréchét-space topology corresponding to uniform convergence in every strip. Let $\mathscr{A} \subset \mathscr{B}$ be the subset of functions g: $\mathbf{C} \to \mathbf{C}$ with $g^{-1}(0)$ equal to Ω or $\mathbf{C}$. Then $\mathscr{A}$ is a closed subset of $\mathscr{B}$.*

Proof: We shal prove that $\mathscr{B} \backslash \mathscr{A}$ is open. That $h \in \mathscr{B} \backslash \mathscr{A}$ means that h: $\mathbf{C} \to \mathbf{C}$ is entire and that there is either a number $\omega \in \Omega$ with $h(\omega) \neq 0$ or a number $z_0 \in \mathbf{C} \backslash \Omega$ with $h(z_0) = 0$. In the first case it is obvious that h is in the interior of $\mathscr{B} \backslash \mathscr{A}$. In the second case there is a disc $D \subset \mathbf{C}$ with center z_0 and positive distance from Ω, and then $|h|$ has infimum $k > 0$ on the boundary of D and according to Rouché's theorem every entire function approximating h with accuracy better than k on the boundary of D has a zero in D and that proves again that h is in the interior of $\mathscr{B} \backslash \mathscr{A}$. That finishes the proof.

We shall start in earnest on our non existence proof. From now on f: $\mathbf{C} \to \mathbf{C}$ is an entire function which is also almost periodic and satisfies that $f^{-1}(0) = \Omega$. Until the end of section 4 we shall use f exclusively as notation for this particular function.

Lemma 1. *To $\varepsilon > 0$, $\delta > 0$, $A > 0$ corresponds $\varepsilon' > 0$ such that every $(\varepsilon', A + \delta)$-translation number τ of f has a corresponding (ε, A)-translation number $\omega \in \Omega$ of f with $|\tau - \omega| \leqq \delta$.*

Proof: Let P denote the closed parallelogram with corners $\pm\frac{1}{2}\omega_1\pm\frac{1}{2}\omega_2$ and $P(\tau)$, $\tau\geqq 0$ the set of numbers $z\in P$ with $|z|\geqq\tau$. With $\tau_0=\frac{1}{2}\min\{|\omega|\mid\omega\in\Omega\setminus\{0\}\}$ we define $\kappa:[0,\tau_0]\to[0,\infty[$ by $\kappa(\tau)=\inf|f|(P(\tau))$. We choose $\tau_1>0$ such that $\tau_1\leqq\delta$, $\tau_1\leqq\tau_0$ and $|f(z_0)-f(z_1)|\leqq\frac{\varepsilon}{2}$ for $z_1,z_2\in S_{A+\delta}$, $|z_2-z_1|\leqq\tau_1$. Next, we choose $\varepsilon'\in]0,\frac{\varepsilon}{2}[$ such that $\varepsilon'<\kappa(\tau_1)$. For the given $(\varepsilon',A+\delta)$-translation number τ of f we choose $\omega\in\Omega$ such that $\tau-\omega\in P$. Since $f(-\omega)=0$, we have $|f(\tau-\omega)|\leqq\varepsilon'$, hence $\tau-\omega\notin P(\tau_1)$, but that implies that $|\tau-\omega|\leqq\tau_1<\delta$. For $z\in S_A$ we have $z-(\tau-\omega)\in S_{A+\delta}$ and we get the estimate

$$
\begin{aligned}
&|f(z+\omega)-f(z)|\\
&\leqq|f(z+\omega)-f(z+\omega-\tau)|+|f(z-(\tau-\omega))-f(z)|\leqq\varepsilon'+\frac{\varepsilon}{2}\leqq\varepsilon,
\end{aligned}
$$

which proves the lemma.

Lemma 2. *For $A>0$ we define $S_A(r)$ as the subset of points of S_A with distance $\geqq r$ from Ω* and we define $k_A:[0,r_0]\to[0,\infty[$ *by $k_A(r)=\inf|f|(S_A(r))$ with r_0 as in the proof of Lemma 1. Then k_A is strictly positive on* $]0,r_0]$.

Proof: We do it indirectly assuming that $k_A(r_1)=0$ for some $r_1\in]0,r_0]$. Then there is a sequence (z_j) with $z_j\in S_A(r_1)$ and $(f(z_j))\to 0$. Let P be the parallelogram from the proof of Lemma 1. We choose (ω_j) with $\omega_j\in\Omega$ such that $z_j-\omega_j\in P$, $j\in\mathbf{N}$. By replacing (z_j) by a convenient subsequence (which we shall still denote (z_j)) we can according to Proposition 1 assume that (f_{ω_j}) converges uniformly in every strip to an entire function $\tilde{f}$: $\mathbf{C}\to\mathbf{C}$, and by the compactness of P we can further assume that $(z_j-\omega_j)\to a\in P$. Since $(z_j-(a+\omega_j))\to 0$ and f is uniformly continuous in every strip, we have also $(f(a+\omega_j))\to 0$, but $(f(a+\omega_j))=(f_{\omega_j}(a))\to\tilde{f}(a)$, hence $\tilde{f}(a)=0$. But $z_j\in S_A(r_1)$ implies that $z_j-\omega_j\in P(r_1)$, and we have $f_{\omega_j}\in\mathcal{A}$, $j\in\mathbf{N}$ and Proposition 2 yields that $\tilde{f}\in\mathcal{A}$, hence $\tilde{f}^{-1}(0)=\Omega$ in contradiction to $\tilde{f}(a)=0$. That proves the lemma.

Lemma 3. *With P as in the proof of Lemma 1 and $b=\max\{|y|\mid z\in P\}$ we define T_f as the closure in $\mathcal{B}$ of $\{f_\tau\mid\tau\in\mathbf{R}\}$. For every $\tilde{f}\in T_f$ there is then an $a\in P$ such that $\tilde{f}_a^{-1}(0)=\Omega$. Further, for $A>0$ and k_A as in Lemma 2 we have $|\tilde{f}(z)|\geqq k_{A+b}(r)$* for every *$z\in S_A$ with distance $\geqq r$ from every zero of $\tilde{f}$.*

Proof: There is a sequence (τ_j) of real numbers such that $(f_{\tau_j})\to\tilde{f}$ uniformly in every strip. We choose (ω_j) with $\omega_j\in\Omega$ and $\tau_j-\omega_j\in P$. By replacing (τ_j) by a convenient subsequence we can assume that $(\tau_j-\omega_j)\to -a$, and since P is symmetric, we have $a\in P$. Since $(\tau_j-(\omega_j-a))\to 0$ and f is uniformly continuous in every strip, we have $(f_{\omega_j-a})\to\tilde{f}$ and $(f_{\omega_j})\to\tilde{f}_a$ uniformly in every strip. Since $f_{\omega_j}\in\mathcal{A}$, the first statement in the lemma follows from Proposition 2. By Lemma 2 it is quite obvious that $|f_{\omega_j}(z)|\geqq k_{A+b}(r)$ for every $z\in S_A$, and the last statement follows by passage to the limit. This ends the proof.

The Fourier exponents of f

Let $g : \mathbf{C} \to \mathbf{C}$ be an arbitrary entire almost periodic function. The function $a : \mathbf{R} \to \mathbf{C}$ defined by
$a(\lambda) = \lim_{T\to\infty} \frac{1}{T}\int_0^T g(x+iy)e^{-2\pi i\lambda(x+iy)}\,dx$ satisfies that $\Lambda_g = \{\lambda \in \mathbf{R} \mid a(\lambda) \neq 0\}$ is at most denumerable so that we have a Fourier series $\Sigma_{\lambda\in\Lambda_g} a(\lambda)e^{2\pi i\lambda z}$. The main theorem in the theory of almost periodic functions states that the Fourier series is summable with sum $g(z)$ and uniformly in every strip. In a more precise form this means that there is a function $k : \Lambda_g \times \mathbf{N} \to [0,1]$ with the following 3 properties:

(1) The set $\{\lambda \in \Lambda_g \mid k(\lambda, n) \neq 0\}$ is finite for every $n \in \mathbf{N}$.
(2) The sequence $(k(\lambda, n))$ tends to 1 for every fixed $\lambda \in \Lambda_g$.
(3) The sequence (s_n) of finite sums
$s_n(z) = \Sigma_{\lambda\in\Lambda_g} k(\lambda, n)a(\lambda)e^{2\pi i\lambda z}$ tends to $g(z)$ uniformly in every strip.

The vector space $\bar{\Lambda}_g \subset \mathbf{R}$ spanned by Λ_g has a base $\gamma = (\gamma_1, \gamma_2, \ldots)$. It may be a finite base $\gamma = (\gamma_1, \ldots, \gamma_m)$, but we shall formulate the following investigations as if the worst happens and only occasionally refer to the rather obvious changes to be made if the basis is finite. By the way, it is easy to see that there is an entire almost periodic function $h : \mathbf{C} \to \mathbf{C}$ such that $g(z)e^{h(z)}$ has the base infinite.

It is very important for our investigations that there is a fundamental relationship between the translation numbers of g and the base γ. This is described in detail in [1] where it is used in the proof of the approximation theorem, and the main points are summarized in [6] p 144-145 and 149-150. Unfortunately, the results are not formulated in terms of the base. We shall reformulate them and add a few remarks in way of proving them.

In this connection we must consider some Diophanthine inequalities of the form $|\gamma\tau - c| \leqq \delta \pmod{n!\mathbf{Z}}$ with $\delta > 0$; $\gamma, c \in \mathbf{R}$, $n \in \mathbf{N}$. That $\tau \in \mathbf{R}$ is solution of the inequality means that there is a $\nu \in \mathbf{Z}$ such that $|\gamma\tau - c - n!\nu| \leqq \delta$. In connection with the base γ we consider the following system of simultaneous Diophanthine inequalities where the second line gives the alternative form for $\gamma = (\gamma_1, \ldots, \gamma_m)$

$$|\gamma_j\tau| \leqq \delta \pmod{n!\mathbf{Z}},\ j = 1, \ldots, n \tag{1}$$
$$|\gamma_j\tau| \leqq \delta \pmod{n!\mathbf{Z}},\ j = 1, \ldots, m.$$

The relationship between γ and the translation numbers of g is given by the proposition:

PROPOSITION 3. *To $\varepsilon > 0$, $A > 0$ correspond $\delta > 0$, $n \in \mathbf{N}$ such that every solution of (1) is an (ε, A)-translation number of g.*

In fact, $\tau \in \mathbf{R}$ is an (ε, A)-translation number of g if it is an $(\frac{\varepsilon}{2}, A)$-translation number of the finite sum s_n which approximates g in S_A with accuracy $\frac{\varepsilon}{4}$. If q is the number of terms in s_n, it follows that τ is an (ε, A)-translation number τ of g, if it is an $(\frac{\varepsilon}{2q}, A)$-translation number of each term $k(\lambda, n)\, a\,(\lambda)e^{2\pi i\lambda z}$, and this will happen, if τ satisfies a set of Diophanthine inequalities $|\lambda_j\tau| \leqq \delta'$ (mod $\mathbf{Z}$), $j = 1, \ldots, q$. We express the λ_j in terms of the γ_j and choose n large enough such that the denominators in the coefficients in these expression are divisors in $n!$ and the proposition follows easily.

There is also a reverse relationship:

Proposition 4. *If $\lambda \in \mathbf{R}$ has the property that to every $A > 0$, $\delta > 0$ exists an $\varepsilon > 0$ such that every (ε, A)-translation number τ og g satisfies the Diophanthine inequality $|\lambda\tau| \leqq \delta$ (mod $\mathbf{Z}$), then $\lambda \in \bar{\Lambda}_g$.*

Proof: If $\lambda \notin \bar{\Lambda}_g$, the numbers $\lambda, \gamma_1, \gamma_2, \ldots$ are independent and Kronecker's theorem tells us that for every $\delta > 0$, $n \in \mathbf{N}$ some solutions of (1) also satisfies the inequality $|\lambda\tau - \frac{1}{2}| \leqq \delta$ (mod $n!\mathbf{Z}$). Hence, for $\delta < \frac{1}{4}$ the condition in the proposition is not satisfied by λ. This proves the proposition.

We shall now return to the hypothetical function f, but first some formulas concerning Ω must be established. For $\omega = \alpha + i\beta \in \Omega$ with $\omega = n_1\omega_1 + n_2\omega_2$, $n_1, n_2 \in \mathbf{Z}$ we have

$$\alpha = n_1\alpha_1 + n_2\alpha_2, \quad \beta = n_1\beta_1 + n_2\beta_2.$$

Eliminating either n_2 or n_1 between these, we get the 2 sets of relations

$$n_2 = -n_1\frac{\beta_1}{\beta_2} + \frac{\beta}{\beta_2}; \; \alpha = n_1\frac{\Delta}{\beta_2} + \frac{\alpha_2}{\beta_2}\beta; \; \omega = n_1\frac{\Delta}{\beta_2} + \frac{\omega_2}{\beta_2}\beta. \tag{2}$$

$$n_1 = -n_2\frac{\beta_2}{\beta_1} + \frac{\beta}{\beta_1}; \; \alpha = -n_2\frac{\Delta}{\beta_1} + \frac{\alpha_1}{\beta_1}\beta; \; \omega = -n_2\frac{\Delta}{\beta_1} + \frac{\omega_1}{\beta_1}\beta. \tag{3}$$

For the function f we shall simply use Λ as notation for the set of Fourier exponents and $\bar{\Lambda}$ for the vector space spanned by Λ. The following lemma tells that $\bar{\Lambda}$ is at least 2-dimensional and, hence, f is not limit periodic.

Lemma 4. $\frac{\beta_1}{\Delta}, \frac{\beta_2}{\Delta} \in \bar{\Lambda}$.

Proof: For $\delta > 0$ we choose $\delta_1 = \frac{\Delta\delta}{|\beta_1|+|\omega_1|}$ and for $A > 0$ we can by Lemma 1 choose $\varepsilon > 0$ such that every $(\varepsilon, A+\delta)$-translation number of τ of f has a corresponding $\omega = \alpha + i\beta = n_1\omega_1 + n_2\omega_2 \in \Omega$ with $|\tau - \omega| \leqq \delta_1$.

In particular, if $\tau \in \mathbf{R}$ we get $|\beta| \leqq \delta_1$ and (3) yields

$$|\tau + n_2\frac{\Delta}{\beta_1}| \leqq \delta_1 + \frac{|\omega_1|}{|\beta_1|}|\beta| \leqq \left(1 + \frac{|\omega_1|}{|\beta_1|}\right)\delta_1 = \frac{\Delta}{|\beta_1|}\delta,$$

hence $|\tau \cdot \frac{\beta_1}{\Delta} + n_2| \leqq \delta$, which is exactly $\left|\frac{\beta_1}{\Delta}\tau\right| \leqq \delta \pmod{\mathbf{Z}}$. Thus, it follows from Proposition 4 that $\frac{\beta_1}{\Delta} \in \bar{\Lambda}$, and that $\frac{\beta_2}{\Delta} \in \bar{\Lambda}$ is proved in the same way.

Since $\frac{\beta_1}{\Delta}, \frac{\beta_2}{\Delta}$ are independent, we can choose the base $\underline{\gamma}$ with $\gamma_1 = \frac{\beta_1}{\Delta}, \gamma_2 = \frac{\beta_2}{\Delta}$ and from now on we shall assume that $\underline{\gamma}$ is chosen like that, and the subspace of $\bar{\Lambda}$ spanned by γ_1 and γ_2 will be called the compulsory subspace.

The spatial extension of f

We shall introduce some functions defined on spaces of pairs $(\underline{x}; y)$ of a finite or infinite sequence $\underline{x} = (x_1, \ldots, x_m)$ or $\underline{x} = (x_1, x_2, \ldots)$ of real numbers, and a real number y. We shall denote the spaces $\mathbf{R}^m \times \mathbf{R}$ or $\mathbf{R}^\infty \times \mathbf{R}$ accordingly and they shall always be organized with the product topology. We shall formulate everything for $\mathbf{R}^\infty \times \mathbf{R}$ only.

If $I \subset \mathbf{R}$ is a bounded interval, we shall call the set $Sl_I = \{(\underline{x}; y) \in \mathbf{R}^\infty \times \mathbf{R} \mid y \in I\}$ the slice corresponding to I, and a slice shall be a set defined in this way by some bounded interval. If $I = [-A, A]$, we shall also write Sl_A for Sl_I. A function $G : \mathbf{R}^\infty \times \mathbf{R} \to \mathbf{C}$ is called limit periodic if it is continuous and satisfies the following condition: To $\varepsilon > 0$, $A > 0$ corresponds $n \in \mathbf{N}$ such that $|G(\underline{x}''; y) - G(\underline{x}'; y)| \leqq \varepsilon$ if $|y| \leqq A$ and $x''_1 - x'_1, \ldots x''_n - x'_n$ are integers divisible by $n!$. It is easy to prove that G is limit periodic if and only if it can be approximated uniformly in any given slice with any given accuracy by a continuous function depending only on finitely many variables $x_1, \ldots, x_m; y$ and with an integral period in $x_1, \ldots, x_m$. However, we shall not use that.

Proposition 5. *Let $G : \mathbf{R}^\infty \times \mathbf{R} \to \mathbf{C}$ be limit periodic and $\underline{\gamma} = (\gamma_1, \gamma_2, \ldots)$ independent. We define $g : \mathbf{C} \to \mathbf{C}$ by $g(z) = G(\underline{\gamma}x; y)$. Then g is almost periodic and Λ_g is contained in the vector space spanned by $\gamma_1, \gamma_2, \ldots$.*

PROOF: Let $\varepsilon > 0, A > 0$ be given. We choose $\delta > 0, n \in \mathbf{N}$ such that $|G(\underline{x}'';y) - G(\underline{x}';y)| \leqq \frac{\varepsilon}{2}$ if either $|x_j'' - x_j'| \leqq \delta, j = 1,\ldots, n$; $|y| \leqq A$ or $x_j'' - x_j'$ for $j = 1,\ldots, n$ is an integer divisible by $n!$ and $|y| \leqq A$. Let τ be a real solution of the inequalities (1). We can choose integers $\nu_1,\ldots, \nu_m$ such that for every $x \in \mathbf{R}$ we have $|\gamma_j(x+\tau) - (\gamma_j x+n!\nu_j)| \leqq \delta, j = 1,\ldots, n$. With $\underline{x}' = (\gamma_1 x + n!\nu_1,\ldots, \gamma_n x + n!\nu_n, \gamma_{n+1}x, \gamma_{n+2}x,\ldots)$ we have the inequalities

$$|g(z+\tau) - G(\underline{x}';y)| \leqq \frac{\varepsilon}{2}\ ; \quad |G(\underline{x}';y) - g(z)| \leqq \frac{\varepsilon}{2},$$

which prove that τ is an (ε, A)-translation number of g. Since the set of real solutions of (1) is relatively dense, this proves that g is almost periodic.

Let λ be a Fourier exponent of g and τ an (ε, A)-translation number of g for some $\varepsilon > 0$, $A > 0$. Then we have

$$a(\lambda)\ (e^{2\pi i\lambda\tau} - 1) = \lim_{T\to\infty} \frac{1}{T}\int_0^T (g(x+\tau) - g(x))e^{-2\pi i\lambda x}dx,$$

which yields the estimate

$$|a(\lambda)|\ |e^{2\pi i\lambda\tau} - 1| \leqq \varepsilon.$$

On the other hand, if $|\lambda\tau - \frac{1}{2}| \leqq \frac{1}{4}$ (mod $\mathbf{Z}$) we obviously have

$$|a(\lambda)|\ |e^{2\pi i\lambda\tau} - 1| \geqq |a(\lambda)|.$$

If λ is not in the vector space spanned by $\gamma_1, \gamma_2,\ldots$, some solutions of (1) will by Kronecker's theorem also satisfy that $|\lambda\tau - \frac{1}{2}| \leqq \frac{1}{4}$ (mod $\mathbf{Z}$), so that they cannot be (ε, A)-translation numbers of g for any $A > 0$ and any $\varepsilon < |a(\gamma)|$. Thus λ is in the space spanned by $\gamma_1\ \gamma_2,\ldots$, and that ends the proof.

With G and g as in Proposition 5 we shall call g the diagonal function of G corresponding to $\underline{\gamma}$ and G a spatial extension of g corresponding to the base $\underline{\gamma}$ of Λ_g. The subspace $C = \{(\underline{\gamma}x;y) \mid x, y \in \mathbf{R}\}$ will be called the $\underline{\gamma}$-diagonal in $\mathbf{R}^\infty \times \mathbf{R}$ and the affine subspaces $C_{\underline{x}} = \{(\underline{x} + \underline{\gamma}x;y) \mid x, y \in \mathbf{R}\}, \underline{x} \in \mathbf{R}^\infty$ will be called the analytic planes in $\mathbf{R}^\infty \times \mathbf{R}$.

PROPOSITION 6. *Let $g : \mathbf{C} \to \mathbf{C}$ be entire and almost periodic, and let $\underline{\gamma} = (\gamma_1, \gamma_2,\ldots)$ be a base for $\bar{\Lambda}_g$. Then g has a uniquely determined spatial extension $G : \mathbf{R}^\infty \times \mathbf{R} \to \mathbf{C}$, and for every $\underline{x} \in \mathbf{R}^\infty$ the function $g_{\underline{x}} : \mathbf{C} \to \mathbf{C}$ defined by $g_{\underline{x}}(z) = G(\underline{x}+\underline{\gamma}x;y)$ is entire and almost periodic and belongs to the closure T_g of $\{g_\tau \mid \tau \in \mathbf{R}\} \subset \mathscr{B}$.*

PROOF: With $X_n = \mathbf{R} \times \mathbf{Z}^n$; $n \in \mathbf{N}$ we define $\psi_n : X_n \to \mathbf{R}^\infty$ by

$$\psi_n(x; \nu_1,\ldots, \nu_n) = (\gamma_1 x + n!\nu_1,\ldots, \gamma_n x + n!\nu_n, \gamma_{n+1}x, \gamma_{n+2}x,\ldots).$$

For each $n \in \mathbf{N}$ the set $M_n = \psi_n(X_n) \times \mathbf{R} \subset \mathbf{R}^\infty \times \mathbf{R}$ is a system of analytic planes. Since ψ_n is injective, we can define $G_n : M_n \to \mathbf{C}$ by

$$G_n(\psi_n(x; \nu_1,\ldots, \nu_n); y) = g(x + iy) = g(z).$$

For every $\underline{x} \in \psi_n(X_n)$ we have $\underline{x} + \underline{\gamma}x \in \psi_n(X_n)$ for every $x \in \mathbf{R}$ so that we can define $g_{n,\underline{x}}$: $C_{\underline{x}} \to \mathbf{C}$ by $g_{n,\underline{x}}(z) = G_n(\underline{x} + \underline{\gamma}x; y)$. Further, there is a $\tau \in \mathbf{R}$ and $\nu_1,\ldots, \nu_n \in \mathbf{Z}$ such that $\underline{x} = \psi_n(\tau; \nu_1,\ldots, \nu_n)$, hence $\underline{x} + \underline{\gamma}x = \psi_n(x + \tau; \nu_1,\ldots, \nu_n)$ and $g_{n,\underline{x}} = g_\tau$. We have thus $g_{n,\underline{x}} \in T_g$.

Let us consider an arbitrary $\underline{x}^\circ \in \mathbf{R}^\infty$ with its corresponding analytic plane $C_{\underline{x}^\circ}$. For $n \in \mathbf{N}$ we define

$$U_n(\underline{x}^\circ) = \{\underline{x} \in \mathbf{R}^\infty \mid |x_j - x_j^\circ| \leqq \frac{1}{2n}, j = 1,\ldots, n\},$$

and the $U_n(\underline{x}^\circ)$ constitute a base for the neighbourhoods of x° in $\mathbf{R}^\infty$. For $q, n \in \mathbf{N}$ we have $\psi_n(X_n) \cap U_q(\underline{x}^\circ) \neq \emptyset$ if the Diophanthine inequalities

$$|\gamma_j x - x_j^\circ| \leqq \frac{1}{2q} \ (\mathrm{mod}\ n!\mathbf{Z}), \quad j = 1,\ldots, q \tag{4}$$

are satisfied by some $x \in \mathbf{R}$. By Kronecker's theorem this is always the case. We have thus proved that the sets $\psi_n(X_n)$, $n \in \mathbf{N}$ are dense in $\mathbf{R}^\infty$. We are interested in the analytic plane $C_{\underline{x}^\circ}$, and by its nth set of neighbour planes we mean the set $V_n(\underline{x}^\circ)$ of planes $C_{\underline{x}}$ with $\underline{x} \in \psi_n(X_n) \cap U_n(\underline{x}^\circ)$. Similarly the set $W_n(\underline{x}^\circ)$ of corresponding entire functions $g_{n,\underline{x}}$ is called the nth set of neighbour functions of $C_{\underline{x}^\circ}$.

For $\varphi \in \mathscr{B}$ and $A > 0$ we define the norm $\|\varphi\|_A = \sup |\varphi|(S_A)$, and the system of norms $\|\cdot\|_A$, $A \in]0,\infty[$ will then induce the Fréchét space topology on $\mathscr{B}$. For a set $\mathscr{M} \subseteq \mathscr{B}$ we can define a generalized diameter by $\mathrm{diam}_A\, \mathscr{M} = \sup\{\|\psi - \varphi\|_A \mid \varphi, \psi \in \mathscr{M}\}$. It is an increasing function of A and it may of course be infinite. For $\underline{x}, \underline{x}' \in \psi_n(X_n) \cap U_n(\underline{x}^\circ)$ we have $x, \tau \in \mathbf{R}$ and $\underline{\nu} = (\nu_1,\ldots, \nu_n)$, $\underline{\nu}' = (\nu_1',\ldots, \nu_n')$, $\nu_j, \nu_j' \in \mathbf{Z}$, $j = 1,\ldots, n$ and $\underline{x} = \psi_n(x,\underline{\nu})$, $\underline{x}' = \psi_n(x + \tau,\underline{\nu}')$. The corresponding functions of $W_n(\underline{x}^\circ)$ are $g_{n,\underline{x}} = g_x$ and $g_{n,\underline{x}'} = g_{x+\tau}$. But x and $x+\tau$ satisfy (4) with $q = n$, hence, τ satisfies (1) with $\delta = \frac{1}{n}$, and Proposition 3 implies that $\mathrm{diam}_A\, W_n(\underline{x}^\circ) \to 0$ for $n \to \infty$ and fixed A, and uniformly for $\underline{x}^\circ \in \mathbf{R}^\infty$.

For $n, q \in \mathbf{N}$ we observe that those $\underline{x} = \psi_n(x,\underline{\nu})$ which have $\nu_1,\ldots, \nu_n$ divisible by $(n+1)\ldots(n + q)$ are also in $\psi_{n+q}(X_n)$ and it follows that some functions of $W_n(\underline{x}^\circ)$ are also in $W_{n+q}(\underline{x}^\circ)$. With $\tilde{W}_n(\underline{x}^\circ) = \bigcup_{q=0}^\infty W_{n+q}(\underline{x}^\circ)$ we can thus conclude that

$$\operatorname{diam}_{\Delta} \tilde{W}_n(\underline{x}^\circ) \leqq 2 \operatorname{diam}_{\Delta} W_n(\underline{x}^\circ).$$

It follows from this that every function of $W_n(\underline{x}^\circ)$ for $n \to \infty$ converges uniformly to the same limit function $g_{\underline{x}^\circ}: \mathbf{C} \to \mathbf{C}$, and it obviously is in T_g. The totality of functions $g_{\underline{x}}$ in every $C_{\underline{x}}$ constitutes a function $G : \mathbf{R}^\infty \times \mathbf{R} \to \mathbf{C}$. It follows immediately from the construction that G is continuous and that $G(\underline{\gamma}x;y) = g(z)$. The limit periodicity of G follows easily from the periodicity of G_n, since $\psi_n(X_n)$ is everywhere dense. This also implies that G is unique and that finishes the proof.

We could have derived it more easily from the approximation theorem, but the proof above underlines certain structural details, which are useful in our investigations.

The hypothetical function $f: \mathbf{C} \to \mathbf{C}$ with the basis $\underline{\gamma} = (\gamma_1, \gamma_1, \ldots)$ where $\gamma_1 = \dfrac{\beta_1}{\Delta}, \gamma_2 = \dfrac{\beta_2}{\Delta}$ has a spatial extension $F : \mathbf{R}^\infty \times \mathbf{R} \to \mathbf{C}$. We shall compute the zeros of F.

Lemma 5. *$F^{-1}(0)$ is the set E given by*

$$E = \{(\alpha_1 t + q_1, \alpha_2 t + q_2, x_3, x_4, \ldots; \Delta t) \mid t, x_3, x_4, \ldots \in \mathbf{R}, q_1, q_2 \in \mathbf{Z}\}.$$

Proof: We shall consider the set E defined in the lemma and we shall prove that it is identical to $F^{-1}(0)$. First, we determine $E \cap C_{\underline{x}^\circ}$ when $C_{\underline{x}^\circ} = \{(\underline{x}^\circ + \underline{\gamma}x; y) \mid x, y \in \mathbf{R}\}$ is an analytic plane. To do that we must determine the sets $(x, y, t; x_3, x_4, \ldots)$ satisfying the equations

$$\begin{gathered} \alpha_1 t + q_1 = x_1^\circ + \gamma_1 x,\ \alpha_2 t + q_2 = x_2^\circ + \gamma_2 x,\ \Delta t = y \\ x_j = x_j^\circ + \gamma_j x,\quad j = 3, 4, \ldots. \end{gathered}$$

With $\gamma_1 = \dfrac{\beta_1}{\Delta}, \gamma_2 = \dfrac{\beta_2}{\Delta}$ the equations in the top row yield

$$x = x_1^\circ\alpha_2 - x_2^\circ\alpha_1 + q_2\alpha_1 - q_1\alpha_2; y = \Delta t = x_1^\circ\beta_2 - x_2^\circ\beta_1 + q_2\beta_1 - q_1\beta_2,$$

and $x_3, x_4, \ldots$ are determined by the equations in the second row. We are not really interested in these additional unknowns. We get

$$z = x + iy = x_1^\circ\omega_2 - x_2^\circ\omega_1 + q_2\omega_1 - q_1\omega_2.$$

We have thus proved that E intersects each analytic plane in $\mathbf{R}^\infty \times \mathbf{R}$ in a translated lattice spanned by ω_1 and ω_2.

It follows from Propositions 2 and 6 that also $F^{-1}(0)$ intersects each analytic plane in $\mathbf{R}^\infty \to \mathbf{R}$ in a translated lattice spanned by ω_1 and ω_2. To prove the theorem we need only that the two lattices in each analytic plane are identical, and that will follow when we have proved that

$$F(\alpha_1 t, \alpha_2 t, x_3, x_4, \ldots, \Delta t) = 0 \text{ for } t, x_3, x_4, \ldots \in \mathbf{R}.$$

By the limit periodicity of F it is enough to prove for every $\delta > 0$, $n \in \mathbf{N}$ that we can find $\omega \in \Omega$, $\omega = \alpha + i\beta$, such that

$$|\gamma_j \alpha - x_j| \leqq \delta \ (\mathrm{mod}\ n!\mathbf{Z}), j = 1, \ldots, n;\ x_1 = \alpha_1 t,\ x_2 = \alpha_2 t,\ |\beta - \Delta t| \leqq \delta. \tag{5}$$

We write $\omega = n_1\omega_1 + n_2\omega_2$, $n_1, n_2 \in \mathbf{Z}$, and (2) and (3) yield

$$\gamma_j \alpha - x_j = \gamma_j \frac{\Delta}{\beta_2} n_1 - x_j' + \varrho_j' = -\gamma_j \frac{\Delta}{\beta_1} n_2 - x_j'' + \varrho_j''$$

with

$$\begin{aligned} x_j' &= x_j - \gamma_j \frac{\alpha_2}{\beta_2} \Delta t, \qquad & \rho_j' &= \gamma_j \frac{\alpha_2}{\beta_2} (\beta - \Delta t), \\ x_j'' &= x_j - \gamma_j \frac{\alpha_1}{\beta_1} \Delta t, \qquad & \rho''_j &= \gamma_j \frac{\alpha_1}{\beta_1} (\beta - \Delta t), \end{aligned} \tag{6}$$

We introduce $\gamma = \max(|\gamma_j \frac{\alpha_k}{\beta_k}| \mid j = 1, \ldots, n;\ k = 1, 2)$ and $\delta' = \frac{\delta}{1+\gamma}$. Then $w = \alpha + i\beta = n_1\omega_1 + n_2\omega_2$ will satisfy (5) if n_1, n_2 satisfy first that $|n_1\beta_1 + n_2\beta_2 - \Delta t| \leqq \delta'$ and second for each $j \in \mathbf{N}$ one of the following Diophanthine inequalities

$$\left|\gamma_j \frac{\Delta}{\beta_2} n_1 - x_j'\right| \leqq \delta' \ (\mathrm{mod}\ n!\mathbf{Z}) \quad \text{or} \quad \left|-\gamma_j \frac{\Delta}{\beta_1} n_2 - x_n''\right| \leqq \delta' \ (\mathrm{mod}\ n!\mathbf{Z}).$$

For $j = 1, 2$ we insert $\gamma_j = \frac{\beta_j}{\Delta}$ and x_j', x_j'' from (6) and we get the inequalities

$$\begin{aligned} &\left|\frac{\beta_1}{\beta_2} n_1 - \frac{\Delta}{\beta_2} t\right| \leqq \delta' \ (\mathrm{mod}\ n!\mathbf{Z}) \quad \text{or} \quad |-n_2| \leqq \delta' \ (\mathrm{mod}\ n!\mathbf{Z}) \\ &|n_1| \leqq \delta' \ (\mathrm{mod}\ n!\mathbf{Z}) \quad \text{or} \quad \left|-\frac{\beta_2}{\beta_1} n_2 + \frac{\Delta}{\beta_1} t\right| \leqq \delta' \ (\mathrm{mod}\ n!\mathbf{Z}). \end{aligned} \tag{7}$$

The second and third of these are satisfied by $n_j = n!\nu_j$, $\nu_j \in \mathbf{Z}$, $j = 1, 2$. The inequality $|n_1\beta_1 + n_2\beta_2 - \Delta t| \leqq \delta'$ becomes

$$|n!\beta_1\nu_1 + n!\beta_2\nu_2 - \Delta t| \leqq \delta'.$$

We observe that the first and the fourth of the inequalities (7) follow from this last inequality and, further, that the last inequality is satisfied by some $\nu_2 \in \mathbf{Z}$, if ν_1 satisfies that $|n!\frac{\beta_1}{\beta_2}\nu_1 - \frac{\Delta}{\beta_2}t| \leqq \frac{\delta'}{|\beta_2|}$ (mod $n!\mathbf{Z}$). Hence, (5) will be satisfied, if $\nu_1 \in \mathbf{Z}$ can be chosen as a solution to the following system of Diophanthine inequalities:

$$\left|n!\frac{\beta_1}{\beta_2}\nu_1 - \frac{\Delta}{\beta_2}t\right| \leqq \frac{\delta'}{|\beta_2|}(\text{mod } n!\mathbf{Z});$$

$$\left|\gamma_j\frac{\Delta}{\beta_2}n!\nu_1 - x_j'\right| \leqq \delta' \ (\text{mod } n!\mathbf{Z}), \quad j = 3, 4, \ldots, n.$$

By a slightly advanced form of Kronecker's theorem we have that this system has integral solutions for all t, x_j', $j = 3, 4, \ldots$, if and only if no linear combination of the coefficients $n!\frac{\beta_1}{\beta_2}$, $n!\gamma_j\frac{\Delta}{\beta_2}$, $j = 3, 4, \ldots$ with integral coefficients has an integral value different from 0. In other words, solutions exist, if

$$q_2 + q_1\frac{\beta_1}{\beta_2} + q_3\gamma_3\frac{\Delta}{\beta_2} + \cdots + q_n\gamma_n\frac{\Delta}{\beta_2} = 0, \quad q_1, \ldots, q_n \in \mathbf{Z}$$

implies that $q_1 = \cdots = q_n = 0$. However, the equation can be written

$$q_1\frac{\beta_1}{\Delta} + q_2\frac{\beta_2}{\Delta} + q_3\gamma_3 + \cdots + q_n\gamma_n = 0,$$

and $\frac{\beta_1}{\Delta}$, $\frac{\beta_2}{\Delta}$ γ_3, γ_4,... are independent. That proves the lemma.

Lemma 6. *Let $\mathbf{R}^2 \times \mathbf{R} \subset \mathbf{R}^\infty \times \mathbf{R}$ be the $(x_1, x_2; y)$-subspace, and $p : \mathbf{R}^\infty \times \mathbf{R} \to \mathbf{R}^2 \times \mathbf{R}$ the projection. Then $E_0 = E \cap (\mathbf{R}^2 \times \mathbf{R})$ is a system of parallel straight lines and $F^{-1}(0) = E = p^{-1}(E_0)$. Further E_0 contains exactly one straight line L_{q_1,q_2} through each point $(q_1, q_2, 0)$ of the unit lattice in $\mathbf{R}^2$. The sets $p^{-1}(L_{q_1,q_2})$ are the components of E and L_{q_1,q_2} intersects the analytic plane C in the point corresponding to $z = q_2\omega_1 - q_1\omega_2$.*

This is nothing more than a reformulation of Lemma 5 supplemented by very few and very elementary computations.

The non existence of f

The spatial extension F of f has a restriction $\varphi : \mathbf{R}^2 \to \mathbf{C}$ defined by $\varphi(x_1, x_2) = F(x_1, x_2, 0, 0, \ldots, 0)$. We know that φ^{-1} (0) is the unit lattice in $\mathbf{R}^2$. The midway net $M \subset \mathbf{R}^2$ is defined as the set of all points (x_1, x_2) with either x_1 or x_2 equal to $\frac{1}{2}$ + some integer. It divides $\mathbf{R}^2$ in unit squares such that φ has one zero in the center of each square.

Lemma 7. $\inf |\varphi|(M) = k > 0$.

Proof: In each analytic plane $C_{\underline{x}} = \{(\underline{x} + \underline{\gamma}x; y) | x, y \in \mathbf{R}\}$ we place discs defined by $x = \alpha + \rho \cos \theta, y = \rho \sin \theta$; $\theta \in \mathbf{R}$, $\varrho \in]0,r[$ for some $r \in]0,r_0[$ (Lemma 2) and for each $\omega = \alpha + i\beta \in \Omega$. The union of all these discs is by Lemma 5 the set of all points of $\mathbf{R}^\infty \times \mathbf{R}$ given by

$$\begin{aligned}&(\alpha_1 t + q_1 + \gamma_1 \rho \cos \theta, \alpha_2 t + q_2 + \gamma_2 \rho \cos \theta,\\ &x_3 + \gamma_3 \rho \cos \theta, x_4 + \gamma_4 \rho \cos \theta, \ldots; \Delta t + \rho \sin \theta)\\ &\theta, t, x_3, x_4, \cdots \in \mathbf{R}, \quad \rho \in [0,r], \quad q_1, q_2 \in \mathbf{Z}.\end{aligned}$$

Let us denote this set E_r and its intersection with $\mathbf{R}^2 \times \mathbf{R}$ by E_r^0. It follows immediately from the expression or from Lemma 6 that $\underline{E}_r = p^{-1}(E_r^0)$ and we have

$$E_r^0 = \left\{ \left(\alpha_1 t + q_1 + \frac{\beta_1 \rho}{\Delta} \cos \theta,\ \alpha_2 t + q_2 + \frac{\beta_2 \rho}{\Delta} \cos \theta;\ \Delta t + \rho \sin \theta \right) \ | \ t, \theta \in \mathbf{R}, \rho \in [0,r], q_1, q_2 \in \mathbf{Z} \right\}.$$

The intersection of E_r^0 with the (x_1, x_2)-plane is

$$\begin{aligned}\tilde{E}_r = \{(&q_1 + \frac{\varrho}{\Delta}(\beta_1 \cos \theta - \alpha_1 \sin \theta),\\ &q_2 + \frac{\rho}{\Delta}(\beta_2 \cos \theta - \alpha_2 \sin \theta)) \mid \theta \in \mathbf{R}, \rho \in [0,r], q_1, q_2 \in \mathbf{Z}\}.\end{aligned}$$

This set consists of elliptic discs with centers in each point of the unit lattice and they are exactly alike and oriented in the same manner. We choose a fixed value of r such that $\tilde{E}_r \cap M = \emptyset$.

We know from Proposition 6 that the restriction of F to an arbitrary analytic plane is a function of T_f. Hence, the lemma follows from Lemma 3 with $A > 0$ chosen arbitrarily. This ends the proof.

Lemma 8. *There is an orientation of the (x_1, x_2)-plane such that the variation of the argument of φ along a small circle around a lattice point and in the direction given by the orientation of the plane is equal to 2π for every point of the unit lattice.*

Proof: For $u \in [0,1]$ and $\underline{\gamma}_u = (\gamma_1, \gamma_2, u\gamma_3, u\gamma_4, \ldots)$ we have the family of planes $C_u = \{\underline{\gamma}_u x; y) \mid x, y \in \mathbf{R}\}$ in $\mathbf{R}^\infty \times \mathbf{R}$ and for each $\omega = \alpha + i\beta \in \Omega$ and $r \in]0, r_0[$ we get a family of circles

$$\Gamma_u = \{(\underline{\gamma}_u(\alpha + r\cos\theta); \beta + r\sin\theta) \mid \theta \in \mathbf{R}\}.$$

From Lemma 6 follows that Γ_u is a continuous family of circles in $\mathbf{R}^\infty \times \mathbf{R} \setminus F^{-1}(0)$. We choose the orientation of each plane C_u such that the angle from the x-axis to the y-axis is $+\frac{\pi}{2}$. Then, the variation of the argument of F along Γ_u has its value independent of u, and since the restriction of F to C_1 is an entire function we conclude that F has its variation of argument equal to 2π along $\Gamma_0 \subset C_0$ for every $\omega \in \Omega$.

From now on we shall consider only the restriction $\tilde{F} : \mathbf{R}^2 \times \mathbf{R} \to \mathbf{C}$ of F defined by $\tilde{F}(x_1, x_2; y) = F(x_1, x_2, 0, 0, \ldots; y)$. We have $C_0 \subset \mathbf{R}^2 \times \mathbf{R}$. It will be convenient to think of $\mathbf{R}^2 \times \mathbf{R}$ as our physical space with the y-axis vertical, and C_0 is then raised as a vertical wall, which divides $\mathbf{R}^2 \times \mathbf{R}$ in two half spaces V_d and V_u such that the lines $L_{q_1 q_2}$ slant downwards in V_d and upwards in V_u. The set E_r^0 from the proof of Lemma 7 is the union of disjoint elliptic cylinders such that each $L_{q_1 q_2}$ is the axis of symmetry of one of them. The circles Γ_0 induce an orientation of each cylinder and we know that the variation of the argument of $\tilde{F}$ along a curve encircling a cylinder once is 2π. In particular this holds for the ellipses, in which E_r^0 intersects the (x_1, x_2)-plane. The orientation of the (x_1, x_2)-plane corresponding to this can be determined in the following way: Start with a circle $\Gamma \subset C_0$ with a diameter in the (x_1, x_2)-plane and oriented according to C_0. Rotate it an angle $\frac{\pi}{2}$ about the horizontal diameter such that its upper half goes into V_d, and it yields the orientation. This finishes the proof.

It must be obvious to everybody that the lemmas 7 and 8 contradict each other, but we must go through the details anyway such that our proof is not left unfinished.

Theorem 1. *Let $\Omega \subset \mathbf{C}$ be a lattice with no real period. Then no entire function $f : \mathbf{C} \to \mathbf{C}$ almost periodic in the direction of the real axis will satisfy that $f^{-1}(0) = \Omega$.*

Proof: If the theorem was false, our hypothetical function f would exist and the lemmas 7 and 8 would hold for some limit periodic function $\varphi : \mathbf{R}^2 \to \mathbf{C}$. Let Γ be the oriented boundary of a square on the midway net M and the length of the sides $n!$ for some large $n \in \mathbf{N}$. Let $v \in \mathbf{R}$ be the variation of the argument of φ along Γ.

By Lemma 8 and the ordinary routine we get $v = 2\pi(n!)^2$.

We choose n large enough such that for every $(x_1, x_2) \in \mathbf{R}^2$

$$|\varphi(x_1, x_2 + n!) - \varphi(x_1, x_2)| \leqq \frac{1}{2}k, \quad |\varphi(x_1 + n!, x_2) - \varphi(x_1, x_2)| \leqq \frac{1}{2}k.$$

Then the variations of the argument of φ along the sides of Γ parallel to the x_1-axis taken together amounts to the variation of the argument of $\frac{\varphi(x_1, x_2 + n!)}{\varphi(x_1, x_2)}$, but this quotient is contained in the angle defined by $|\arg z| \leqq \frac{\pi}{6}$, hence the variation of the argument along these sides amount to at most $\frac{\pi}{3}$. The same holds for the two other sides, and since the variation of the argument along Γ is an integer multiplied by 2π, we can conclude that $v = 0$.

This proves the theorem.

Application of Weierstrass' σ-function

We shall use the notations Ω, ω_1, ω_2, α_1, β_1, β_2, Δ as before. With $\Omega' = \Omega \setminus \{0\}$ Weierstrass' σ-function is defined by

$$\sigma(z) = z\Pi_{\omega\in\Omega'} \left(1 - \frac{z}{\omega}\right) e^{\frac{z}{\omega} + \frac{z^2}{2\omega^2}}.$$

It is an entire function of order 2 with $\sigma^{-1}(0) = \Omega$, and though it is not periodic, there are constants $\eta_1, \eta_2 \in \mathbf{C}$ satisfying

$$\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i \tag{8}$$

such that σ has the periodicity property

$$\sigma(z + \omega_j) = -e^{\eta_j (z+\frac{1}{2}\omega_j)}\sigma(z), \quad j = 1, 2$$

and for $\omega = n_1\omega_1 + n_2\omega_2$ with $\eta = n_1\eta_1 + n_2\eta_2$ we have generally

$$\sigma(z + \omega) = (-1)^{n_1n_2+n_1+n_2}e^{\eta(z+\frac{1}{2}\omega)}\sigma(z).$$

From this follows that the function $f_\omega : \mathbf{C} \to \mathbf{C}$ defined by

$$f_\omega(z) = e^{-\frac{\eta}{2\omega}z^2}\sigma(z)$$

has period 2ω and even ω if n_1 and n_2 are even. We remark that f_ω depends only on the direction of ω not on ω itself. We supply (3) with a corresponding formula for η so that we have

$$\omega = -n_2\frac{\Delta}{\beta_1} + \frac{\omega_1}{\beta_1}\beta, \quad \eta = -n_2\frac{\eta_1\beta_2 - \eta_2\beta_1}{\beta_1} + \frac{\eta_1}{\beta_1}\beta,$$

and if we let $\omega \to \infty$ while $\beta \to 0$, the ratio $\frac{\eta}{\omega}$ will tend to

$$\gamma = \frac{\eta_1\beta_2 - \eta_2\beta_1}{\Delta}$$

and $f_\omega(z)$ tends to the limit

$$f(z) = e^{-\frac{\gamma}{2}z^2}\,\sigma(z),$$

an entire function with $f^{-1}(0) = \Omega$ and obviously satisfying

$$f(z+\omega) = \pm\, e^{(\eta-\gamma\omega)\,(z+\frac{1}{2}\omega)}\, f(z).$$

We do some computation

$$\begin{aligned}\eta - \gamma\omega &= \frac{1}{\Delta\beta_1}(-\eta_2\Delta(\eta_1\beta_2 - \eta_2\beta_1) + \Delta\eta_1\beta - (\eta_1\beta_2 - \eta_2\beta_1)\,(-\eta_2\Delta + \omega_1\beta))\\ &= \frac{1}{\Delta\beta_1}((\alpha_1\beta_2 - \alpha_2\beta_1)\eta_1\beta - (\eta_1\beta_2 - \eta_2\beta_1)\omega_1\beta),\end{aligned}$$

but since

$$\alpha_1\beta_2 - \alpha_2\beta_1 = (\omega_1 - i\beta_1)\beta_2 - (\omega_2 - i\beta_2)\beta_1 = \omega_1\beta_2 - \omega_2\beta_1,$$

it reduces to

$$\eta - \gamma\omega = \frac{1}{\Delta}(-\eta_1\omega_2 + \eta_2\omega_1)\beta = -2\pi i\,\frac{\beta}{\Delta}$$

by (8). Thus we have

$$f(z+\omega) = \pm\, e^{-2\pi\frac{\beta}{\Delta}\,(z+\frac{1}{2}\omega)} f(z)$$

and with $\omega = \alpha + i\beta$, $\quad z = iy$ this implies

$$|f(z+\omega)| = e^{-2\pi\frac{\beta}{\Delta}\,(y+\frac{1}{2}\beta)}|f(z)|.$$

We know from Theorem 1 that $f: \mathbf{C} \to \mathbf{C}$ is not almost periodic. Nevertheless, we have the following

THEOREM 2. *The function* $|f| : \mathbf{C} \to \mathbf{R}$ *is almost periodic.*

Proof: We shall first prove that f is bounded in every strip, hence, we consider S_A, $A > 0$. We choose $L > 0$ such that every interval on $\mathbf{R}$ of length L contains a number α for which there is a $\beta \in [-1, 1]$ with $\alpha + i\beta = \omega \in \Omega$. We define $K = \max|f|([0, L] \times [-A-1, A+1])$ and for an arbitrary $z \in S_A$ we can then find $\omega = \alpha + i\beta \in \Omega$ with $|\beta| \leqq 1$ and $x - \alpha \in [0, L]$, hence $z - \omega \in [0, L] \times [-A-1, A+1]$. It follows that

$$|f(z)| \leqq K\, e^{2\pi \frac{|\beta|}{A}(A+1)},$$

and this proves that f is bounded in S_A, i.e. in every strip. Since f is entire, this implies that f is uniformly continuous in every strip.

Let $\varepsilon > 0$ be given. We choose $\delta_1 \in]0, 1]$ such that for every $z \in S_A$ and every $\omega = \alpha + i\beta$ with $|\beta| \leqq \delta$, we have

$$|f(z+\omega) - f(z+\alpha)| \leqq \frac{1}{2}\varepsilon.$$

With the K introduced above we choose $\delta_2 > 0$ such that for $|\beta| < \delta_2$

$$\left|e^{2\pi\frac{\beta}{A}(A+\frac{1}{2}|\beta|)} - 1\right| \leqq \frac{\varepsilon}{2K}\, e^{-\frac{2\pi}{A}(A+\frac{3}{2})},$$

and with $\delta = \min(\delta_1, \delta_2)$ for $|\beta| \leqq \delta$ we have

$$\big||f(z+\omega)| - |f(z)|\big| \leqq \left|e^{2\pi\frac{\beta}{A}(A+\frac{1}{2}|\beta|)} - 1\right| |f(z)| \leqq \frac{1}{2}\varepsilon,$$

and together with the preceding inequality this proves that α is an (ε, A)-translation number of $|f|$, and that proves the theorem.

Theorem 3. *With* $\overline{\Omega} = \{\omega \in \mathbf{C} \mid \overline{\omega} \in \Omega\}$ *there is an entire almost periodic function* $g : \mathbf{C} \to \mathbf{C}$ *of order* 2 *and with* $g^{-1}(0) = \Omega \cup \overline{\Omega}$ *such that the elements of* $\Omega \cup \overline{\Omega}$ *are simple zeros, except* 0 *which is double.*

Proof: We define g by $g(z) = f(z)\overline{f(\bar{z})}$ and $g : \mathbf{C} \to \mathbf{C}$ is entire of order 2 and $g^{-1}(0)$ is as claimed in the theorem. By the multiplication theorem $|g| : \mathbf{C} \to \mathbf{R}$ is almost periodic, hence g is bounded in every strip. For $z = x \in \mathbf{R}$ we have $g(x) = |g|(x)$, hence, g is almost periodic on the real axis. But this implies that g is almost periodic in every strip ([2], p. 253). This proves the theorem.

A more general entire almost periodic function with $g^{-1}(0) \supset \Omega$ could be defined by $g_a(z) = f(z+a)\overline{f(\bar{z}+\bar{a})}$ for some $a \in \mathbf{C}$.

Bibliography

The few facts about elliptic functions used in the paper can be found in most comprehensive texts. The author recommends the presentation in Hurwitz-Courant: Funktionentheorie, because it introduces the relevant functions at an early stage.

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Matematisk Institut
Universitetsparken 5
DK-2100 København
Denmark

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